A_∞-CATEGORIES, FUKAYA CATEGORIES AND GENTLE ALGEBRAS

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Abstract. These are notes from a mini-course titled "A geometric model for the derived category of a gentle algebra" given in Trondheim from June 10 to 13 2025. They are currently incomplete, as they only treat the part of the course that dealt with A_∞-categories. They will be updated soon to include a chapter on Fukaya categories of surfaces.

Contents

This is an extended version of notes from a mini-course titled *A geometric model for the derived category of a gentle algebra* that I gave in Trondheim from June 10 to June 13 of 2025. In writing these notes, I tried to keep the spirit of the four 45-minutes talks that made up the course, while filling in gaps or adding technical details. My aim was to introduce Fukaya categories of surfaces, which use the machinery of A_∞-categories, to an audience of researchers in the field of representation theory of finite-dimensional associative algebras. For this reason, I assumed familiarity with some homological constructions: projective modules, complexes of projective modules, the homotopy category $K^b(\operatorname{proj}\Lambda)$ and the bounded derived category $\mathcal{D}^b(\operatorname{mod}\Lambda)$. Some familiarity with dg categories will help understand some of the examples and analogies in the text but is not strictly necessary.

The core of these notes is an introduction to A_∞-categories and twisted complexes. This is followed by an application: Haiden, Katzarkov and Kontsevich's construction of the partially wrapped Fukaya category of a surface with stops. The notes end with an explanation of why this construction gives a powerful tool to study the derived category of a gentle algebra.

A roadmap: from complexes of projectives to twisted complexes

The framework of A_∞-categories and twisted complexes is a vast generalization of that which arises from complexes of projective modules. Before we embark on formal definitions, let us recall a step by step construction of $K^b(\operatorname{proj}\Lambda)$, with an idea on how each step will be generalized by A_∞-categories. Fix a field K.

Step 1. We start with an algebra $\Lambda = KQ/I$ given by a quiver with relations (Q, I). We view (Q, I) as a K-linear category whose objects are the vertices of Q and whose morphism

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space from i to j is the space of linear combination of paths from i to j modulo the relations in I; in other words, $\operatorname{Hom}(i,j) = e_j\Lambda e_i$, where e_i and e_j are the paths of length zero at vertices i and j, respectively.

Equivalently, this is the category whose objects are the indecomposable projective right Λ-modules $P_i = e_i\Lambda$ and the morphisms are the morphisms of Λ-modules. Denote this category by $\operatorname{ind}\operatorname{proj}\Lambda$.

→ We will replace (Q,I) (or equivalently, $\operatorname{ind}\operatorname{proj}\Lambda$) with an A_∞-category $\mathcal{A}$ where spaces of morphisms $\operatorname{Hom}(i,j)$ are replaced with graded vector spaces, and composition of morphisms is replaced with an infinite family of *higher multiplication* μ^n.

Step 2. From $\operatorname{ind}\operatorname{proj}\Lambda$, we first form the category of all finitely generated projective modules $\operatorname{proj}\Lambda$ by allowing finite direct sums, and we then construct the category $C^b(\operatorname{proj}\Lambda)$ of complexes of finitely generated projective modules.

→ From the A_∞-category $\mathcal{A}$, we will form the category $\operatorname{add}\mathbb{Z}\mathcal{A}$ by formally allowing finite direct sums and shifts of objects, and we replace the notion of a complex of modules by that of a *twisted complex* over $\mathcal{A}$. We will then replace $C^b(\operatorname{proj}\Lambda)$ with the A_∞-category $\operatorname{Tw}\mathcal{A}$ of twisted complexes over $\mathcal{A}$.

Step 3. The triangulated category $K^b(\operatorname{proj}\Lambda)$ is obtained from $C^b(\operatorname{proj}\Lambda)$ by taking the quotient by the ideal of null-homotopic morphisms of complexes. The shift of complexes and mapping cones of morphisms induce the shift functor and the distinguished triangles of its triangulated structure.

→ The *cohomology category* $H^0\operatorname{Tw}\mathcal{A}$ has a natural triangulated category structure given by an appropriate definition of shift and mapping cones.

1. A_∞-CATEGORIES

The notion of an A_∞-category was introduced by Stasheff in 1963 [Sta63].[1] Here are some more recent references where one can learn about A_∞-categories.

- A book by Paul Seidel [Sei08]. The first chapter is an excellent and detailed introduction to A_∞categories and twisted complexes over them.
- Two introductory papers by Bernhard Keller [Kel02, Kel06].
- A book by Raf Bocklandt with applications to Fukaya categories of surfaces [Boc21].
- A non-commutative-geometric take by Maxim Kontsevich and Yan Soibelman [KS09]

We fix a field K.[2]

Definition 1.1. An A_∞-category $\mathcal{A}$ is given by the following data.

- A class $\operatorname{Ob}\mathcal{A}$ of *objects.*
- For all objects X and Y, a $\mathbb{Z}$-graded K-vector space $\operatorname{Hom}_{\mathcal{A}}(X,Y) = \bigoplus_{i\in\mathbb{Z}}\operatorname{Hom}^i_{\mathcal{A}}(X,Y)$ of *morphisms.* If $a \in \operatorname{Hom}_{\mathcal{A}}(X,Y)$, we write $|a| = i$ and $\|a\| = |a| - 1$.
- (Higher multiplications.) For all integers $n \geq 1$, for all objects $X_0, X_1, \ldots, X_n$, a multilinear map

$$\mu^n_{\mathcal{A}} : \operatorname{Hom}_{\mathcal{A}}(X_{n-1},X_n) \times \cdots \times \operatorname{Hom}_{\mathcal{A}}(X_1,X_2) \times \operatorname{Hom}_{\mathcal{A}}(X_0,X_1) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X_0,X_n)$$

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[1] In the cited paper, Stasheff defined A_n-algebras, of which A_∞-categories are an immediate generalization.

[2] We could replace K with a commutative ring R.

of degree $n-2$. These must satisfy the following "quadratic relations", which we will call the *Stasheff relations*: for all $n \geq 1$ and all homogeneous morphisms $a_i \in \mathrm{Hom}_{\mathcal{A}}(X_{i-1}, X_i)$ $(i = 1, \ldots, n)$,

$$\sum_{[\ell, k+1] \subset [n,1]} (-1)^{\|a_k\| + \ldots + \|a_1\|} \mu_{\mathcal{A}}^{k+n-\ell+1} \Big(a_n, \ldots, a_{\ell+1}, \mu_{\mathcal{A}}^{\ell-k}(a_\ell, \ldots, a_{k+1}), a_k, \ldots, a_1 \Big) = 0,$$

where $[b, a]$ denotes the discrete interval $\{b, b-1, \ldots, a+1, a\}$.

The A_∞-category $\mathcal{A}$ is *strictly unital* if, moreover, for every object X, there exists an *identity morphism* $1_X \in \mathrm{Hom}^0_{\mathcal{A}}(X, X)$ such that

- for all $a \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, we have $\mu^2_{\mathcal{A}}(a, 1_X) = a$;
- for all homogeneous $b \in \mathrm{Hom}_{\mathcal{A}}(W, X)$, we have $\mu^2_{\mathcal{A}}(1_X, b) = (-1)^{|b|} b$;
- for all $n \geq 3$, we have $\mu^n_{\mathcal{A}}(\ldots, 1_X, \ldots) = 0$.

When there is no risk of confusion, we will omit the index and write μ^n instead of $\mu^n_{\mathcal{A}}$. Since the n in μ^n denotes the number of entries, we will also omit it when no confusion can arise. With these choices of notations, the Stasheff relations become

$$\sum_{[\ell, k+1] \subset [n,1]} (-1)^{\|a_k\| + \ldots + \|a_1\|} \mu \Big(a_n, \ldots, a_{\ell+1}, \mu(a_\ell, \ldots, a_{k+1}), a_k, \ldots, a_1 \Big) = 0.$$

Example 1.2. Define $\mathcal{A}$ as follows. It has four objects X_1, X_2, X_3 and X_4. For all i (taken modulo 4), the space $\mathrm{Hom}_{\mathcal{A}}(X_i, X_{i+1})$ is one-dimensional and generated by a morphism a_i. The degree of a_i is given by $|a_1| = |a_2| = |a_3| = 0$ and $|a_4| = 2$. Moreover, the space $\mathrm{Hom}_{\mathcal{A}}(X_i, X_i)$ is generated by a morphism 1_{X_i} which satisfies the axioms of an identity morphism in a strictly unital A_∞-category. All other morphism spaces vanish. The higher multiplications are defined by:

- $\mu^1 = 0$;
- $\mu^2(a_{i+1}, a_i) = 0$ for all i (modulo 4);
- $\mu^4(a_{i+3}, a_{i+2}, a_{i+1}, a_i) = 1_{X_i}$ for all i (modulo 4).

The above are extended by multilinearity; all other μ^n are zero. It is an exercise to check that $\mathcal{A}$ is a strictly unital A_∞-category.

We think of this A_∞-category as a "quiver with higher multiplications"; in particular, we depict it as follows.

$$\begin{tikzcd}
X_1 \arrow[r, "|a_1|=0"] & X_2 \arrow[d, "|a_2|=0"] \\
X_4 \arrow[u, "|a_4|=2"] & X_3 \arrow[l, "|a_3|=0"]
\end{tikzcd}$$

It is worth spelling out the Stasheff relations for small values of n.

For $n = 1$. The relation is $\mu^1\Big(\mu^1(a)\Big) = 0$. For this reason, we think of μ^1 as a *differential.*

For $n = 2$. The relation is

$$\mu^1\Big(\mu^2(a_2, a_2)\Big) + \mu^2\Big(a_2, \mu^1(a_1)\Big) + (-1)^{|a_1|}\mu^2\Big(\mu^1(a_2), a_1\Big) = 0.$$

If we think of μ^1 as a differential and of μ^2 as a multiplication, then this is akin to the *Leibnitz rule* encountered in dg categories.

The definition of an A_∞-category specializes to K-linear categories or dg categories for certain choices of higher multiplications.

Example 1.3. If $\mu^1 = 0$ and $\mu^k = 0$ for $k \geq 3$, then μ^2 is an associative multiplication law, up to a sign. More precisely, setting $a \circ b = (-1)^{|b|}\mu^2(a,b)$, we recover the definition of a graded K-linear category. If, moreover, all morphisms have degree zero, then we recover the definition of a K-linear category.

Example 1.4. If $\mu^k = 0$ for $k \geq 3$, then setting $da = (-1)^{|a|}\mu^1(a)$ and $a \circ b = (-1)^{|b|}\mu^2(a,b)$ recovers the definition of a differential graded (dg) category.

An important observation is that, in any A_∞-category $\mathcal{A}$ and any objects X, Y in $\mathcal{A}$, the pair $\Big(\operatorname{Hom}_{\mathcal{A}}(X,Y), \mu^1\Big)$ is a usual complex of K-vector spaces.

Definition 1.5. Let $\mathcal{A}$ be a strictly unital A_∞-category. Its *cohomological category* is the graded K-linear category $H\mathcal{A}$ whose objects are the same as those of $\mathcal{A}$ and whose morphism spaces are the graded K-vector spaces

$$\operatorname{Hom}_{H\mathcal{A}}(X,Y) := H^*\Big(\operatorname{Hom}_{\mathcal{A}}(X,Y), \mu^1\Big),$$

with composition defined on homogeneous morphisms by $[a] \circ [b] = (-1)^{|b|}\big[\mu^2(a,b)\big]$, where $[a]$ is the class of a in cohomology.

Its *zeroth cohomology category* $H^0\mathcal{A}$ is the K-linear subcategory of $H\mathcal{A}$ obtained by keeping only morphisms of degree 0.

2. Twisted complexes

In the previous section, we have introduced the notion of A_∞-categories as a generalization of that of K-linear categories; in an example, we have seen how this generalizes the notion of a quiver with relations (Q, I), viewed as an additive category. In this section, we wish to generalize the notion of a bounded complex of finitely-generated projective modules. This will be done by introducing *twisted complexes.*

But first, let us review and slightly reformulate the usual definition of a complex of projective modules. Let (Q, I) be a quiver with relations and let $\Lambda = KQ/I$. A *bounded complex* X of finitely generated projective modules is an object of the form

$$X = \qquad \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots$$

where

- the P^i are finitely-generated projective Λ-modules;
- the d^i are morphisms of Λ-modules;
- for any $i \in \mathbb{Z}$, we have that $d^{i+1} \circ d^i = 0$; and
- $P^i = 0$ for $i \ll 0$ and $i \gg 0$.

An equivalent way to describe X, and one which we will use to generalize to twisted complexes, is as follows.

- For ear cohomological degree $i \in \mathbb{Z}$, we put a finitely projective module P^i. This projective module is, up to isomorphism, a direct sum of indecomposable summands of Λ. Denote the category of such objects by $\operatorname{add}\Lambda$.

- To get P^i in cohomological degree i, we need to shift it, so what we have actually $P^i[-i]$. The data of all $P^i[-i]$ is encoded in one direct sum $\bigoplus_{i\in\mathbb{Z}} P^i[-i]$. Denote the category of such objects by $\operatorname{add}\mathbb{Z}\Lambda$.
- The data of all differentials d^i can be encoded in a single endomorphism d of $\bigoplus_{i\in\mathbb{Z}} P^i[-i]$ given in matrix form by

$$d = \begin{bmatrix} \ddots & & & & & \\ \ddots & 0 & & & & \\ & d^{i-1} & 0 & & & \\ & & d^i & 0 & & \\ & & & d^{i+1} & \ddots & \\ & & & & \ddots & \end{bmatrix}.$$

  This endomorphism d has degree 1 and is such that $d \circ d = 0$. Its matrix form is also strictly lower triangular.

We can thus write the complex X as $X = \Big(\bigoplus_{i\in\mathbb{Z}} P^i[-i], d\Big)$.

The definition of a twisted complex over an A_∞-category $\mathcal{A}$ is a generalization of the above point of view. We will do this in several steps, which we first present informally.

- The role of indecomposable projective modules is played by the objects of $\mathcal{A}$.
- We need to introduce formal "shifts" and "direct sums" of objects of $\mathcal{A}$ in order to generalize $\bigoplus_{i\in\mathbb{Z}} P^i[-i]$. This will lead to the definition of the A_∞-category $\operatorname{add}\mathbb{Z}\mathcal{A}$.
- A twisted complex is then an object of $\operatorname{add}\mathbb{Z}\mathcal{A}$ together with a morphism δ of degree 1 satisfying a generalization of $d \circ d = 0$.

The objects of $\operatorname{add}\mathbb{Z}\mathcal{A}$ are direct sums of objects of the form $X[n]$, where X is an object of $\mathcal{A}$ and n is an integer. We encode this with a "tensorial" notation. The object $X[0]$ is encoded by $K \otimes X$, where K is viewed as a K-vector space of dimension 1 (and where $\otimes$ should really be seen as just a symbol). The object $X[n]$ is encoded by $K[n] \otimes X$, where $K[n]$ is viewed as a graded vector space ($[n]$ is then a usual degree shift). A direct sum $X[m] \oplus X[n]$ is encoded by $\big(K[m] \oplus K[n]\big) \otimes X$. In full generality, a direct sum of shifts of X is encoded by $V_X \otimes X$, where V_X is a graded K-vector space. Thus an general object of $\operatorname{add}\mathbb{Z}\mathcal{A}$ would have the form $\bigoplus_{X\in\mathcal{A}} V_X \otimes X$.

We are now ready to give a formal definition of $\operatorname{add}\mathbb{Z}\mathcal{A}$.

Definition 2.1. Let $\mathcal{A}$ be a strictly unital A_∞-category. We define $\operatorname{add}\mathbb{Z}\mathcal{A}$ as follows.

- Its objects are symbols of the form $\bigoplus_{X\in\mathcal{A}} V_X \otimes X$, where each V_X is a finite-dimensional graded K-vector space such that $\sum_{X\in\mathcal{A}} \dim V_X$ is finite.
- Morphism spaces are defined by

$$\operatorname{Hom}_{\operatorname{add}\mathbb{Z}\mathcal{A}}(V_X \otimes X, V_Y \otimes Y) := \operatorname{Hom}(V_X, V_Y) \otimes_K \operatorname{Hom}_{\mathcal{A}}(X, Y),$$

  extended linearly to direct sums.
- Higher multiplications are defined by

$$\mu^k_{\operatorname{add}\mathbb{Z}\mathcal{A}}(\phi_k \otimes a_k, \ldots, \phi_1 \otimes a_1) := (-1)^{\sum_{i<j} |\phi_i|\cdot\|a_j\|} \phi_k \circ \cdots \circ \phi_1 \otimes \mu^k_{\mathcal{A}}(a_k, \ldots, a_1).$$

One checks readily (if tediously) that $\operatorname{add}\mathbb{Z}\mathcal{A}$ is a strictly unital A_∞-category.

We now get to the main definition of this section.

Definition 2.2. Let $\mathcal{A}$ be a strictly unital A_∞-category. A *twisted complex* over $\mathcal{A}$ is a pair (V, δ), where

- $V = \bigoplus_{X \in \mathcal{A}} V_X \otimes X$ is an object of $\operatorname{add} \mathbb{Z}\mathcal{A}$;
- $\delta \in \operatorname{Hom}^1_{\operatorname{add} \mathbb{Z}\mathcal{A}}(V, V)$ is its *differential.* It must satisfy that
  - there is an ordering of the direct factors of $\bigoplus_{X \in \mathcal{A}} V_X \otimes X$ in which the matrix form of δ is strictly lower triangular, and
  - $\sum_{k \geq 1} \mu^k_{\operatorname{add} \mathbb{Z}\mathcal{A}}(\delta, \ldots, \delta) = 0.$

Let $\operatorname{Tw} \mathcal{A}$ be the A_∞-category of twisted complexes, defined as follows.

- Its objects are all twisted complexes over $\mathcal{A}$.
- Morphism spaces are defined by
$$\operatorname{Hom}_{\operatorname{Tw} \mathcal{A}}\Big((V, \delta), (W, \epsilon)\Big) := \operatorname{Hom}_{\operatorname{add} \mathbb{Z}\mathcal{A}}(V, W).$$
- Higher multiplications are defined by
$$\mu^k_{\operatorname{Tw} \mathcal{A}}(a_k, \ldots, a_1) := \sum \mu_{\operatorname{add} \mathbb{Z}\mathcal{A}}(\delta, \ldots, \delta, a_k, \delta, \ldots, \delta, a_{k-1}, \delta, \ldots\ldots, \delta, a_1, \delta, \ldots, \delta),$$
where the sum is taken over all possible ways of inserting finitely many δ in between the a_i (including cases where no δ is inserted between the a_i).

It is again a (quite tedious) exercise to check that $\operatorname{Tw} \mathcal{A}$ is a strictly unital A_∞-category. Another exercise is to check that the condition that δ be strictly lower triangular implies that the sum in the definition of higher multiplications has only finitely many non-zero terms.

Notation 2.3.
- For the objects of $\operatorname{Tw} \mathcal{A}$, we will denote $K[n] \otimes X$ by $X[n]$.
- We denote by s^{n-m} the shifted identity functor from $K[m]$ to $K[n]$. It is a morphism of degree $m - n$.

  Using these notations, we get a morphism $s^{n-m} \otimes 1_X : X[m] \to X[n]$.

Example 2.4. We keep $\mathcal{A}$ as in Example 1.2. The following is a twisted complex.

$$X = \left(X_1[0] \oplus X_2[-1] \oplus X_3[-2], \delta = \begin{bmatrix} 0 & 0 & 0 \\ s^{-1} \otimes a_1 & 0 & 0 \\ 0 & s^{-1} \otimes a_2 & 0 \end{bmatrix} \right)$$

We depict this as follows.

$$\begin{tikzcd}
X_1[0] \arrow[r, "s^{-1}\otimes a_1"] & X_2[-1] \arrow[r, "s^{-1}\otimes a_2"] & X_3[-2]
\end{tikzcd}$$

An interesting exercise is to check that, in $H^0 \mathcal{A}$, the object X above is isomorphic to $X_4[-2]$. The isomorphism from $X_4[-2]$ to X is given by $s^2 \otimes a_4 : X_4[-2] \to X_1[0]$; we depict this as follows.

$$\begin{tikzcd}
 & & X_4[-2] \arrow[dll, "s^2\otimes a_4"'] \\
X_1[0] \arrow[r, "s^{-1}\otimes a_1"] & X_2[-1] \arrow[r, "s^{-1}\otimes a_2"] & X_3[-2]
\end{tikzcd}$$

Its inverse is given by $s^0 \otimes a_3 : X_3[-2] \to X_4[-2]$, depicted as follows.

$$\begin{tikzcd}
X_1[0] \arrow[r, "s^{-1}\otimes a_1"] & X_2[-1] \arrow[r, "s^{-1}\otimes a_2"] & X_3[-2] \arrow[d, "s^0\otimes a_3"] \\
 & & X_4[-2]
\end{tikzcd}$$

The proof that the composition of these two morphisms yields the identity morphism uses the non-vanishing of $\mu^4_{\mathcal{A}}$.

The notation in the above example can be used more generally: for instance, if (V, δ) is such that $V = X_1[d_1] \oplus \ldots \oplus X_n[d_n]$, then we write

$$\begin{tikzcd}
X_1[d_1] \arrow[r, "\delta_{21}"] \arrow[rr, bend left, "\delta_{31}"] \arrow[rrrr, bend left, "\delta_{n1}"] & X_2[d_2] \arrow[r, "\delta_{32}"] \arrow[rrr, bend left, "\delta_{n2}"] & X_3[d_3] \arrow[r, "\delta_{43}"] \arrow[rr, bend left, "\delta_{n3}"] & {} \arrow[r, "\delta_{n,n-1}"] & X_n[d_n]
\end{tikzcd}$$

Example 2.5. Take (Q, I) a quiver with relations, viewed a K-linear category. Let $\mathcal{A}$ be the corresponding A_∞-category with $\mu^1 = 0$ and $\mu^k = 0$ for $k \geq 3$, see Example 1.3. Then $H^0 \operatorname{Tw} \mathcal{A}$ is equivalent to $K^b(\operatorname{proj} KQ/I)$.

Example 2.6. Take Λ a finite-dimensional K-algebra, and let $\mathcal{A}$ be the A_∞-category with only one object X such that $\operatorname{Hom}_{\mathcal{A}}(X, X) = \Lambda$. Then $H^0 \operatorname{Tw} \mathcal{A}$ is equivalent to the homotopy category of complexes of *free* finitely generated Λ-modules.

One of the fundamental results in the theory is the following.

Theorem 2.7. *Let $\mathcal{A}$ be a strictly unital A_∞-category. Then $H^0 \operatorname{Tw} \mathcal{A}$ is canonically a triangulated category.*

To define the triangulated structure of $H^0 \operatorname{Tw} \mathcal{A}$, one needs to define a shift functor, mapping cones, and distinguished triangles. This is done, for example, in [Sei08, Chapter I.3]

3. A_∞-FUNCTORS

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be strictly unital A_∞-categories. A (strictly unital) *strict A_∞-functor* $F : \mathcal{A} \to \mathcal{B}$ is defined by

- a map $F : \operatorname{Ob} \mathcal{A} \to \operatorname{Ob} \mathcal{B}$;
- for all objects X and Y, a graded morphism $F : \operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{B}}(X, Y)$ of degree 0 which "commutes with all higher multiplications", that is,
$$F(\mu^n_{\mathcal{A}}(a_n, \ldots, a_1)) = \mu^n_{\mathcal{B}}(F(a_n), \ldots, F(a_1)),$$
for all n and composable $a_1, \ldots, a_n$;
- for all objects X, $F(1_X) = 1_{FX}$.

Example 3.2. The inclusion functor $\mathcal{A} \to \operatorname{Tw} \mathcal{A}$ which sends an object X to $X[0]$ and a morphism $a : X \to Y$ to $s^0 \otimes a : X[0] \to Y[0]$ is a strict A_∞-functors.

The notion of a strict A_∞-functor is usually too rigid for one to truly work with A_∞-categories. It is a special case of a more general definition of an A_∞-functor. Although we will only use strict A_∞-functors in these notes, we give the general definition for completeness.

Definition 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be strictly unital A_∞-categories. A (strictly unital) *A_∞-functor* $F : \mathcal{A} \to \mathcal{B}$ is defined by

- a map $F : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$;
- for all integers $n \geq 1$ and for all objects $X_0, \ldots, X_n$, a linear map of degree $1 - n$
  $$F^n : \operatorname{Hom}_{\mathcal{A}}(X_{n-1}, X_n) \otimes \cdots \otimes \operatorname{Hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(FX_0, FX_n)$$
  such that
  $$\sum \mu_{\mathcal{B}}\Big(\ldots, F^{m_2}(a_{m_1+m_2}, \ldots, a_{m_1+1}), F^{m_1}(a_{m_1}, \ldots, a_1)\Big) =$$
  $$= \sum_{k,\ell} (-1)^{\|a_k\|+\ldots+\|a_1\|} F^{n-\ell+1}\Big(a_n, \ldots, a_{k+\ell+1}, \mu_{\mathcal{A}}^{\ell}(a_{k+\ell}, \ldots, a_{k+1}), a_k, \ldots, a_1\Big),$$
  where the sum on the left-hand side is taken over all partitions $m_1 + m_2 + \ldots = n$;
- $F^1(1_X) = 1_{FX}$ for all objects X;
- $F^n(\ldots, 1_X, \ldots) = 0$ for all $n \geq 2$ and all objects X.

A strict A_∞-functor is then just an A_∞-functor for which F^n vanishes for $n \geq 2$. A basic property of A_∞-functors is that they induce functors of usual categories in cohomology.

Proposition 3.4. *Any A_∞-functor $F : \mathcal{A} \to \mathcal{B}$ induces*

- *an A_∞-fucntor $\operatorname{Tw} F : \operatorname{Tw}\mathcal{A} \to \operatorname{Tw}\mathcal{B}$;*
- *a functor of K-linear graded categories $HF : H\mathcal{A} \to H\mathcal{B}$;*
- *a functor of K-linear categories $H^0F : H^0\mathcal{A} \to H^0\mathcal{B}$.*

Moreover, the induced functor $H^0 \operatorname{Tw} F : H^0 \operatorname{Tw}\mathcal{A} \to H^0 \operatorname{Tw}\mathcal{B}$ is triangulated.

Definition 3.5. Let $F : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor.

- We say that F is a *quasi-equivalence* if $H^0F : H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence of categories.
- We say that F is a *Morita equivalence* if the induced functor $\operatorname{Tw} F : \operatorname{Tw}\mathcal{A} \to \operatorname{Tw}\mathcal{B}$ is a quasi-equivalence.

Theorem 3.6. *For any strictly unital A_∞-category $\mathcal{A}$, the inclusion $\mathcal{A} \to \operatorname{Tw}\mathcal{A}$ is a Morita equivalence.*

Example 3.7. We continue with Example 1.2. Let $\mathcal{B}$ be the full subcategory with objects X_1, X_2 and X_3. Then the inclusion $\mathcal{B} \to \mathcal{A}$ is a Morita equivalence. To see this, one can observe that $H^0 \operatorname{Tw}\mathcal{A}$ is a triangulated category generated by X_1, X_2, X_3 and X_4, and use Example 2.4 to see that X_4 is in the subcategory generated by X_1, X_2 and X_3.

Definition 3.8. Let $\mathcal{A}$ be a strictly unital A_∞-category. A *generator* of $\operatorname{Tw}\mathcal{A}$ is a full subcategory $\mathcal{B}$ of $\operatorname{Tw}\mathcal{A}$ such that the inclusion $\mathcal{B} \to \operatorname{Tw}\mathcal{A}$ is a Morita equivalence. It is *formal* if there is a quasi-equivalence $H^0\mathcal{B} \to \mathcal{B}$, where $H^0\mathcal{B}$ is viewed as an A_∞-category with vanishing μ^1 and μ^k for $k \geq 3$.

If $\mathcal{B}$ is a generator of $\operatorname{Tw}\mathcal{A}$, then the composition $H^0\mathcal{B} \to \mathcal{B} \to \operatorname{Tw}\mathcal{A}$ is a Morita equivalence. But since $H^0\mathcal{B}$ has vanishing μ^1 and μ^k for $k \geq 3$, we can view it as a dg category with vanishing differential, so that the (idempotent closure) of the triangulated category $H^0 \operatorname{Tw}\mathcal{B}$ is the perfect derived category of $H^0\mathcal{B}$.

Example 3.9. We continue Example 1.2, with $\mathcal{B}$ as in Example 3.7. Then $\mathcal{B}$ is a formal generator. In particular, $H^0 \operatorname{Tw} \mathcal{A}$ is equivalent to the perfect derived category of $H^0\mathcal{B}$, which is in turn equivalent to the perfect derived category of the algebra given by the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3$$

and relation $a_2 a_1 = 0$. This is an example of a *gentle algebra*.

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