Derived Categories, Surfaces and Localization

Catégories dérivées, surfaces et localisation

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Thèse soutenue à Versailles, le 15 janvier 2026, par

Pierre Bodin

Composition du jury

Membres du jury avec voix délibérative

Claire AMIOT Professeure, Université Grenoble-Alpes	Présidente
Raf BOCKLANDT Associate Professor, University of Amsterdam	Rapporteur & Examinateur
Emmanuel WAGNER Professeur, Université Paris-Cité	Rapporteur & Examinateur
Emily CLIFF Professeure adjointe, Université de Sherbrooke	Examinatrice
Ana-Maria CASTRAVET Professeure, Université de Versailles Saint-Quentin-en-Yvelines	Examinatrice

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Titre: Catégories dérivées, surfaces et localisation
Mots clés: Théorie des représentations, Algèbre homologique, Combinatoire des surfaces, Catégorie A_∞, Catégorie de Fukaya topologique, Recollement

Résumé: Cette thèse étudie la localisation des catégories de Fukaya topologique de surfaces marquées graduées, telles que définies par Haiden-Katzarkov-Kontsevich, en une collection d'objets sphériques compatibles. Géométriquement, une telle collection correspond à une famille de courbes fermées simples graduées disjointes sur la surface. Nous chercherons à motiver l'idée que la catégorie localisée peut jouer le rôle d'une catégorie de Fukaya topologique pour la surface singulière obtenue en contractant les courbes fermées simples correspondantes.

Dans le premier chapitre, nous rappelons des résultats classiques de la théorie des représentations des algèbres aimables et introduisons les concepts qui nous seront utiles pour les catégories A_∞, avant de présenter la construction de la catégorie de Fukaya topologique d'une surface marquée graduée. Certains aspects du modèle géométrique pour la catégorie dérivée bornée d'une algèbre aimable sont également présentés.

Dans le deuxième chapitre, nous définissons une classe d'algèbres données par carquois et relations, que nous appelons algèbres aimables contractées. En se basant sur un résultat similaire pour les algèbres aimables, nous montrons que les carquois gradués contractés sont en bijection avec les surfaces marquées avec singularités coniques munies d'une dissection graduée admissible simple. Nous étudions ensuite un enrichissement différentiel gradué (DG) de la localisation de la catégorie dérivée parfaite d'une algèbre aimable par une sous-catégorie engendrée par une collection d'objets sphériques compatibles. Nous montrons que chaque algèbre aimable contractée peut être réalisée comme anneau d'endomorphismes d'un générateur formel d'un tel quotient DG. Afin de montrer la formalité, nous utilisons une suite spectrale sur les espaces de morphismes d'un quotient DG et nous décrivons sa première page en toute généralité. Enfin, nous déduisons d'un résultat de Gyenge le fait que la catégorie dérivée d'une algèbre aimable contractée prend place dans un recollement similaire à celui obtenu par Chang-Jin-Schroll pour les localisations en une collection d'arcs compatibles.

Le troisième chapitre est essentiellement une généralisation des résultats du second, en élargissant la classe des générateurs formels obtenus pour les localisations de catégories de Fukaya topologiques par des objets sphériques compatibles. Plus précisément, une généralisation de la classe des algèbres aimables contractées est donnée, ainsi qu'une généralisation de la notion de dissection admissible d'une surface marquée avec singularités coniques. La bijection entre ces deux classes d'objets, établie au chapitre deux, est alors décrite dans cette nouvelle généralité. Nous étudions ensuite un enrichissement A_∞ du quotient triangulé et montrons que chaque algèbre aimable contractée peut être réalisée comme anneau d'endomorphismes d'un générateur formel. Le calcul de la formalité est effectué cette fois-ci au moyen d'un transfert d'homotopie de structure A_∞.

Cette construction nous permet d'associer à une surface graduée marquée avec singularités coniques S munie d'une dissection admissible A, une catégorie $\mathcal{F}_A(S)$ qui engendre la catégorie dérivée parfaite d'une algèbre aimable contractée. Une conséquence immédiate des équivalences données dans le cas lisse par Haiden-Katzarkov-Kontsevich est que la classe d'équivalence de Morita de $\mathcal{F}_A(S)$ est indépendante de A.

Nous donnons au passage deux bases pour les algèbres aimables contractées, l'une étant obtenue par une application du lemme du diamant de Bergman. La dernière section donne un exemple de catégorie triangulée non-Krull-Schmidt contenant deux objets bousculants ayant un nombre différent de composantes directes indécomposables.

Title: Derived Categories, Surfaces and Localization

Keywords: Representation theory, Homological algebra, Combinatorics of surfaces, A_∞-category, Topological Fukaya category, Recollement

Abstract: This doctoral dissertation studies the localization of the topological Fukaya category of a graded marked surface, as defined by Haiden-Katzarkov-Kontsevich, at a collection of compatible spherical objects. Geometrically, such a collection corresponds to a set of disjoint graded simple closed curves on the surface. We seek to motivate the idea that the localized category can play the role of a topological Fukaya category for the singular surface obtained by contracting the corresponding simple closed curves.

In the first chapter we recall classical results of the representation theory of gentle algebras and introduce the relevant concepts regarding A_∞-categories, before presenting the construction of the topological Fukaya category of a graded marked surface. Some aspects of the geometric model for the bounded derived category of a gentle algebra are also presented.

In the second chapter we define a class of algebras given by quivers with relations, that we call pinched gentle algebras. Based on a similar result for gentle algebras, we show that graded pinched gentle quivers are in one-to-one correspondence with marked surfaces with conical singularities endowed with a graded simple admissible dissection. We then study a differential graded (DG) enhancement of the localization of the perfect derived category of a gentle algebra by a subcategory generated by a collection of compatible spherical objects. We show that each pinched gentle algebra arises as the endomorphism ring of a formal generator of such a DG quotient. In order to show the formality, we use a spectral sequence on the morphism spaces of a DG quotient and describe its first page in a general way. Finally, using a result of Gyenge, we deduce that the derived category of a pinched gentle algebra sits in a recollement similar to the one obtained by Chang-Jin-Schroll for the localization at a collection of compatible arcs.

The third chapter essentially generalizes the results of the second one by enlarging the class of formal generators obtained for localizations of topological Fukaya categories by compatible spherical objects. More precisely, a generalization of the class of pinched gentle algebras is given, as well as a generalization of the notion of admissible dissection of a marked surface with conical singularities. The one-to-one correspondence between these objects, given in the second chapter, is then described in this new generality. We then study a A_∞-enhancement of the triangulated quotient and show that each pinched gentle algebra arises as the endomorphism ring of a formal generator. The computation of the formality is done this time by means of a homotopy transfer of A_∞-structure.

This construction allows us to associate to a graded marked surfaces with conical singularities S, endowed with an admissible dissection A, a category $\mathcal{F}_A(S)$ which generates the perfect derived category of a pinched gentle algebra. An immediate consequence of the equivalences given in the smooth case by Haiden-Katzarkov-Kontsevich is that the Morita equivalence class of $\mathcal{F}_A(S)$ is independent of A.

We give two bases for pinched gentle algebras along the way, one of which is obtained using Bergman's Diamond Lemma. The last section provides an example of a non-Krull-Schmidt triangulated category containing two silting objects having a different number of indecomposable summands.

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Contents

Introduction en français

0.1 Contexte

L'objectif de cette thèse est d'obtenir une description simplifiée du quotient de la catégorie de Fukaya topologique d'une surface marquée, par une collection d'objets sphériques. Géométriquement ces objets correspondent à des courbes fermées simples disjointes sur la surface, et prendre le quotient revient à contracter chacune de ces courbes en un point. L'exemple d'une courbe fermée simple sur un cylindre est représenté sur la Figure 0.1.1.

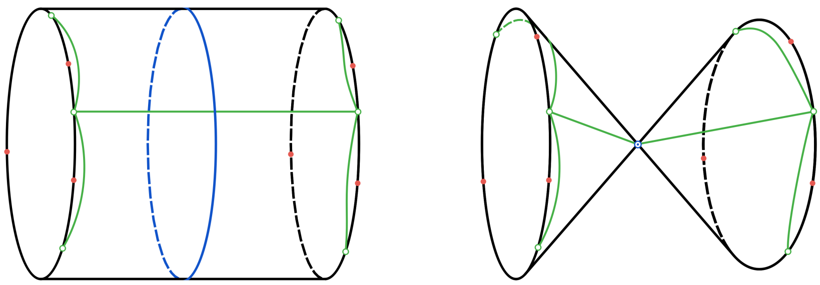


Figure 0.1.1: Une courbe fermée simple (bleue) sur une surface marquée, et la surface marquée avec singularité conique obtenue par contraction.

Dans [Jef22], l'auteur considère une fibration symplectique $f : X \to \mathbb{C}$ d'hypersurfaces, où seule $f^{-1}(0)$ est singulière. Il y définit la catégorie de Fukaya enroulée de la fibre singulière comme étant le quotient de la catégorie de Fukaya enroulée d'une fibre proche, par une sous-catégorie induite par le foncteur de monodromie. Notre travail peut être vu comme l'étude d'un analogue algébrique de cette construction dans le cas des surfaces, avec pour objectif de donner une expression explicite du quotient comme catégorie dérivée parfaite d'une algèbre donnée par générateur et relations. Cela nous conduira à travailler avec la classe des *algèbres aimables*.

Depuis leur introduction [AH81, AH82, AS87], les algèbres aimables se sont avérées être liées à de nombreux autres domaines des mathématiques. Leur théorie des représentations a été largement étudiée, par exemple dans [WW85, GP68, BR87, Cra89, Kra91]. Ayant pour origine la théorie des algèbres amassées provenant de surfaces triangulées [FG09, FST08], des modèles géométriques pour les algèbres aimables ont ensuite été développés dans des directions diverses [Lab09, ABCP10, BCS21].

Les algèbres aimables sont également apparues dans le contexte de la symétrie miroir homologique en tant qu'algèbres d'endomorphismes de générateurs formels de catégories de Fukaya topologiques de surfaces graduées marquées [HKK17]. Réciproquement, dans des travaux tels que [LP20, OPS25, PPP19, BCS21], à toute algèbre aimable graduée est associée une surface marquée munie d'une dissection admissible graduée. Dans le cas où l'algèbre est homologiquement lisse, [LP20] donne une équivalence entre la catégorie dérivée bornée de l'algèbre et la catégorie de Fukaya topologique de la surface marquée graduée associée. En se basant sur la description de la catégorie dérivée bornée de l'algèbre aimable donnée dans [BM03, BD05, ALP16, ÇPS19, ÇPS21], il est montré dans [OPS25] que la surface marquée avec dissection admissible sert de modèle pour les objets indécomposables, pour les morphismes irréductibles et leurs cônes, ainsi que pour les triangles d'Auslander-Reiten.

Ces modèles géométriques ont depuis eu de nombreuses applications dans l'étude des catégories dérivées bornées d'algèbres aimables graduées. Par exemple, en généralisant un invariant dérivé numérique donné dans [AG08] (voir aussi [BH07]), ils ont permis d'établir un invariant dérivé complet pour ces algèbres [LP20, APS23, Opp19, JSW25, Opp25].

Plus généralement, de nombreuses opérations algébriques sur la catégorie dérivée trouvent une interprétation géométrique au niveau de la surface modèle. Ce fut le cas par exemple dans [CS23, CJS23, JSW25] où il est montré, entre autres, que la mutation bousculante correspond à un retournement de la diagonale dans un quadrilatère. Un autre exemple est donné dans [Opp19], avec la description des twists sphériques de la catégorie dérivée comme twists de Dehn sur la surface.

L'étude de la localisation par un arc de bord a été faite dans [HKK17], et utilise la localisation des catégories A_∞ strictement unitaires introduites dans [LO06]. Plus généralement, une étude de la localisation par une collection admissible d'arcs a été faite dans [CS23, CJS23]. Il y est montré que la surface obtenue en découpant le long des arcs donne lieu à un recollement au niveau des catégories dérivées.

Dans cette thèse, nous étudions la localisation par une collection de courbes fermées simples disjointes. Une telle courbe correspond à un objet de la catégorie de Fukaya, qui s'avère être sphérique au sens de [ST01]. Introduits dans un premier temps pour construire des actions catégoriques du groupe des tresses (voir aussi [KS02, GTW17, AL17, Opp19] pour d'autres actions similaires), les objets sphériques jouent désormais un rôle important dans l'étude des catégories triangulées.

Un de nos objectifs sera de motiver l'idée qu'un tel quotient par un objet sphérique peut jouer le rôle d'une catégorie de Fukaya topologique pour la surface singulière obtenue en contractant la courbe fermée simple correspondante.

0.2 Résultats principaux

Dans cette section le corps de base est supposé être de caractéristique zéro.

0.2.1 Résultats principaux du Chapitre 2

Dans le Chapitre 2, nous étudions la localisation de la catégorie dérivée $\mathcal{D}(\Lambda)$ d'une algèbre aimable graduée Λ, par un objet de bande sphérique. Nous montrons que cette localisation est équivalente à la catégorie dérivée d'une algèbre $\Lambda_{(\alpha,\beta)}$ (Théorème 2.1.3), et cela nous conduit à définir une classe d'algèbres données par carquois et relations, que nous appelons *algèbres aimables contractées* (Définition 2.1.6).

La notion de surface marquée avec singularités coniques est introduite dans la Définition 2.4.8 et la Remarque 2.4.9, comme étant l'espace topologique obtenu par la contraction d'une collection de courbes fermées simples disjointes sur une surface marquée lisse. La notion de dissection admissible simple graduée sur une surface marquée avec singularités coniques y est également définie et nous montrons la proposition suivante:

Proposition 0.2.1 *(Proposition 2.4.10) Les carquois aimables contractés gradués sont en bijection avec les surfaces marquées avec singularités coniques munies d'une dissection admissible simple graduée.*

La Définition 2.1.9 donne une procédure pour obtenir une nouvelle algèbre aimable contractée $\Lambda_{(\alpha,\beta)}$ à partir d'une algèbre aimable contractée Λ contenant un sous-carquois d'un certain type, appelée Kronecker acyclique gradué et notée (α, β). De plus, chaque Kronecker acyclique gradué donne lieu à un objet de $per(\Lambda)$, appelé objet de bande supporté sur (α, β). Le résultat principal est le suivant:

Théorème 0.2.2 *(Théorèmes 2.1.11 et 2.1.10) Soit Λ une aglèbre aimable contractée contenant un Kronecker acyclic gradué (α, β). Il y a un recollement:*

$$\mathcal{D}(\Lambda_{(\alpha,\beta)}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathcal{D}(\Lambda) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathcal{D}(K[x]/(x^2)) \quad ,$$

où x est de degré 1. Cela induit une équivalence triangulée:

$$per(\Lambda)/thick(B) \simeq per(\Lambda_{(\alpha,\beta)}),$$

où B est un objet de bande supporté sur (α, β).

Soit Λ une algèbre aimable contractée graduée, et S_Λ la surface marquée avec singularités coniques associée munie d'une dissection admissible simple graduée. On peut appliquer le Théorème 0.2.2 pour décrire la localisation de $per(\Lambda)$ par une courbe fermée simple γ de nombre d'enroulement zéro sur S_Λ (ne passant pas par une singularité), de la manière suivante.

En appliquant le Théorème 0.2.2 un nombre fini de fois, $per(\Lambda)$ peut être réalisé comme un quotient de la catégorie dérivée parfaite d'une algèbre aimable $\hat{\Lambda}$. Les équivalences dérivées pour $\hat{\Lambda}$ données par [HKK17](Proposition 3.2) nous permettent de montrer la Proposition 2.4.3 qui dit que Λ est dérivée équivalente à une algèbre aimable contractée graduée Λ' pour laquelle γ correspond à un objet de bande supporté sur un Kronecker acyclique gradué (α, β). On peut alors appliquer le Théorème 0.2.2.

Le Théorème 0.2.2 est prouvé de la manière suivante. Nous réalisons d'abord $\mathcal{D}(\Lambda)$ comme la catégorie dérivée $\mathcal{D}(\mathcal{A})$ d'une catégorie DG $\mathcal{A}$ contenant une sous-catégorie strictement pleine et formelle $\mathcal{B}$ d'homologie $K[x]/(x^2)$, avant d'appliquer le Théorème 1.4.13. La Proposition 2.5.5 montre que le quotient DG $\mathcal{A}/\mathcal{B}$ est formel et donne une description explicite de sa catégorie cohomologique. Le calcul est basé sur l'utilisation d'une suite spectrale sur les espaces de morphismes d'un quotient DG, dont la première page est décrite en toute généralité dans la Proposition 2.3.4. On note en passant que le calcul des autres pages, effectué dans la Proposition 2.5.10, est rendu possible par le fait que l'objet B est sphérique, et bénéficie donc d'une homologie simple. Enfin, nous montrons dans le Lemme 2.6.2 que $H^*(\mathcal{A}/\mathcal{B})$ est Morita équivalente à l'algèbre aimable contractée $\Lambda_{(\alpha,\beta)}$.

0.2.2 Résultats principaux du Chapitre 3

Le Chapitre 3 généralise les résultats obtenus dans le Chapitre 2 en élargissant la classe des algèbres aimables contractées et en les réalisant comme générateurs formels de localisations, par des objets sphériques, de catégories de Fukaya topologiques de surfaces marquées graduées.

La stratégie utilisée ici diffère de celle utilisée dans le Chapitre 2 par le fait que nous travaillons cette fois avec un enrichissement A_∞ du quotient triangulé. Cependant, même si les preuves sont différentes, le calcul de l'homologie d'un générateur formel est similaire (on pourra comparer la définition de Ψ dans la preuve de la Proposition 2.5.10, et celle de T dans la Notation 3.5.6).

Premièrement, la classe des algèbres aimables contractées est définie (Définition 3.3.2) en généralisant celle donnée en Définition 2.1.6, et la notion de dissection admissible sur une surface marquée avec singularités coniques est introduite, généralisant celle de dissection admissible simple de la Définition 2.4.8.

Ensuite, la Sous-section 3.3.3 associe un carquois aimable contracté avec relations (Q, I) à chaque dissection admissible graduée A d'une surface marquée avec singularités coniques S. La catégorie $\mathcal{F}_A(S)$ est alors définie comme étant la catégorie de chemins $\mathcal{P}(Q, I)$ dont les objets sont Q_0 et les espaces de morphismes sont $\mathcal{P}(Q, I)(i, k) = e_k(KQ/\langle I\rangle)e_i$ (Définition 3.3.6).

Supposons maintenant que S est une surface marquée ayant une singularité conique obtenue en contractant une courbe fermée simple γ de nombre d'enroulement zéro, sur une surface marquée graduée lisse $\hat{S}$. Soit A une dissection admissible sur S. La Sous-section 3.3.4 donne une manière naturelle de relever A en une dissection admissible $\hat{A}$ de $\hat{S}$. Notre théorème principal est le suivant:

Théorème 0.2.3 *(Theorem 3.5.3) Il y a une équivalence de Morita:*

$$\mathcal{F}_A(S) \to \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B}).$$

Ici $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$ désigne le quotient A_∞ (voir Définition 1.3.11) de la catégorie de Fukaya topologique $\mathcal{F}(\hat{S})$ de $\hat{S}$ par la sous-catégorie pleine $\mathcal{B}$ supportée sur les objets isomorphes à des éléments de $thick(B)$ après passage en homologie zéro, B désignant un objet de bande sphérique associé à γ (voir Sous-section 3.3.5).

Puisque la Proposition 3.2 de [HKK17] dit que la classe d'équivalence de Morita de $\mathcal{F}_{\hat{A}}(\hat{S})$ est indépendante de $\hat{A}$, une conséquence immédiate du Théorème 0.2.3 est que ce résultat est aussi vrai pour $\mathcal{F}_A(S)$ (pour des choix de dissections admissibles graduées A sur S induisant une même gradation sur $\hat{S}$). De plus, des calculs non inclus dans cette thèse suggèrent que ce théorème peut être utilisé pour classifier les objets indécomposables de $\mathcal{F}_A(S)^{tr}$ en terme de courbes graduées sur S, en analogie avec [HKK17](Théorème 4.3). Inspiré par [HKK17], nous appelons $\mathcal{F}(S) := Tw\mathcal{F}_A(S)$ la catégorie de Fukaya topologique de S.

En itérant la preuve du théorème 0.2.3, il est possible d'obtenir un enoncé portant sur des surfaces marquées avec plusieurs singularités coniques.

Le théorème 0.2.3 est démontré de la manière suivante. D'après le Lemme 3.5.2, la dissection admissible A (vue comme un sous-ensemble de $\hat{A}$) génère $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$. Cela permet de définir une sous-catégorie pleine $\mathcal{A}$ de $\mathcal{F}(\hat{S})$ qui induit un générateur de $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$ en prenant un quotient A_∞

de la forme $\mathcal{D}(\mathcal{A}|B)$. Afin de simplifier la description de $\mathcal{D}(\mathcal{A}|B)$ nous calculons dans la Proposition 3.4.8 un modèle minimal pour $\mathcal{A}$, noté $H^*\mathcal{A}$. Cela induit un nouveau générateur $\mathcal{D} := \mathcal{D}(H^*\mathcal{A}|B)$ de $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$, qui d'après la Proposition 3.5.10 est formel. Comme ce fut le cas dans le Chapitre 2, le calcul de $H^*\mathcal{D}$ est rendu possible par le fait que B est un objet sphérique et qu'il bénéficie donc d'une homologie simple (voir la preuve de Proposition 3.5.7). Finalement, le Théorème 3.5.11 décrit la catégorie cohomologique de $\mathcal{D}$ comme étant la catégorie des chemins sur le carquois aimable contracté associé à A, c'est-à-dire comme étant $\mathcal{F}_A(S)$. Le diagramme de la Sous-section 3.1.1 résume la situation.

Une première base pour les algèbres aimables contractées est donnée dans la Remarque 3.5.14. La Proposition 3.5.15 en donne une deuxième, obtenue par une application du lemme du diamant de Bergman. Elle est ensuite utilisée dans la preuve du Théorème 3.5.11.

La dernière Section 3.6 donne un exemple de catégorie de Fukaya topologique (triangulée) d'une surface marquée contractée, qui contient deux objets bousculants ayant un nombre différent de composantes directes indécomposables. Cela montre que [Al12](Corollaire 2.28) n'est en général pas vrai lorsque la catégorie triangulée n'est pas supposée être Krull-Schmidt.

Chapter 1

Introduction

1.1 Context

The aim of this thesis is to describe the partially wrapped Fukaya category of a graded marked surface after taking a quotient by certain spherical objects. Geometrically, such an object corresponds to a closed curve on the surface, and taking the quotient will amount to contracting the curve.

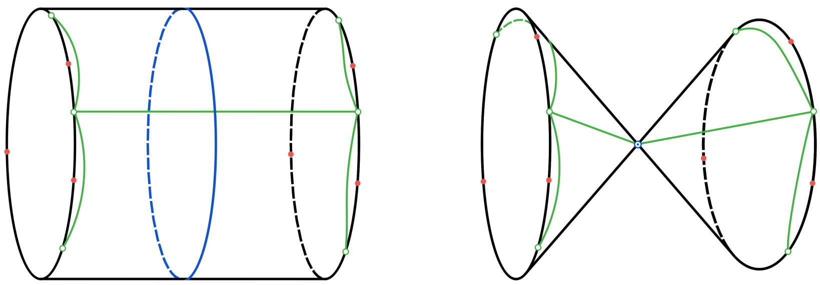


Figure 1.1.1: A marked surface with a simple closed curve (blue), and its marked surface with conical singularities obtained by contraction.

In [Jef22], the author considers families $f : X \to \mathbb{C}$ of symplectic manifolds with a singular fiber over 0, and defines the wrapped Fukaya category of the singular fiber to be a certain localization of the Fukaya category of a nearby fiber. Our work can be seen as an algebraic analogous construction in which we seek an explicit description of the localization. We will do this through the use of *gentle algebras*.

Since their introduction [AH81, AH82, AS87], gentle algebras were found to be linked to many other areas of mathematics. Their representation theory has been widely studied, in particular in [WW85, GP68, BR87, Cra89, Kra91]. Originating in the theory of cluster algebras from triangulated surfaces [FG09, FST08], geometric models for gentle algebras have been developed in numerous directions [Lab09, ABCP10, BCS21].

Gentle algebras also arose in the context of Homological Miror Symmetry as endomorphism algebras of formal generators in partially wrapped Fukaya categories of graded marked surfaces [HKK17].

Conversely, in works such as [LP20, OPS25, PPP19, BCS21], to any graded gentle algebra is associated a graded marked surface with graded admissible dissection. In the homologically smooth case, [LP20] showed that the bounded derived category of the algebra in equivalent to the partially wrapped Fukaya category of the associated graded marked surface. Moreover, based on the description of the bounded derived category of gentle algebra given in [BM03, BD05, ALP16, ÇPS19, ÇPS21], it is shown in [OPS25] that the marked surface with admissible dissection serves as a model for the indecomposable objects, the irreducible morphisms and their cones, and the Auslander-Reiten triangles.

These geometric models have since been fruitful for the study of the bounded derived categories of gentle algebras. For example, this led to a complete derived invariant for graded gentle algebras [LP20, APS23, Opp19, JSW25, Opp25], generalizing a numerical derived invariant of [AG08](see also [BH07]).

More generally, many algebraic operations on the derived category find an interpretation in terms of transformations at the level of the surface model. For example, [CS23, CJS23, JSW25] illustrated the usefulness of this correspondence in the study of the silting theory of the derived category. In particular, silting mutation corresponds to the changing of graded arcs and in some cases to the flipping of diagonals in a quadrilateral. Another example is the correspondence established in [Opp19] between spherical twists of the derived category and Dehn twists on the surface.

The study of the localization at a boundary arc was done in [HKK17], relying on the localization of strictly unital A_∞-categories introduced in [LO06]. A more general study of such localization at an admissible collection of arcs was done in [CS23, CJS23]. It is shown that the surface obtained by cutting along the arcs, and what the authors call the surface obtained by supporting along the arcs, give rise to a recollement at the level of the derived categories.

In this thesis we study the localization at a simple closed curve. Such a curve corresponds to a spherical object in the Fukaya category. Spherical objects were introduced in [ST01] in order to construct categorical actions of the braid group. Since then, they have been shown to play an important role in the study of triangulated categories, and further similar braid group actions were investigated (see for instance [ST01, KS02, GTW17, AL17, Opp19] for more details).

One of our objectives is to motivate the fact that such a quotient by a spherical object can play the role of a topological Fukaya category for the singular surface obtained by contracting the corresponding simple closed curve.

The rest of this chapter is structured as follows. Section 1.2 introduces the class of gentle algebras and states the classification theorem of [BM03] for indecomposable objects in the bounded derived category of a gentle algebra. In Section 1.3, basic definitions and properties regarding A_∞-categories are introduced. In particular, a special case of the A_∞-localization defined in [LO06] is presented, as well as the notion of homotopy transfer. Subsection 1.3.2 translates some of the concepts introduced for A_∞-categories to the context of differential graded categories. The Section 1.4 is devoted to the topological Fukaya categories of a graded marked surfaces, as defined in [HKK17]. After introducing the relevant definitions and properties, we describe how to obtain a gentle quiver from a full formal arc system. Reciprocally, we explain in subsection 1.4.2 how to construct a graded marked surface from a graded gentle algebra. We also present some aspects of the geometric model given by the surface for the bounded derived category of the gentle algebra, following [OPS25]. The next subsection, 1.4.3, states the recollement of [CJS23] obtained by cutting the surface along a collection of disjoint arcs. We finally present in Section 1.5 the results obtained in Chapter 2 and 3.

1.2 Gentle algebras

In this section we recall the definition of a gentle algebra as well as the classification of indecomposable objects in its bounded derived category. Throughout we work with ungraded algebras. The classification in the graded case is given in [OPS25](Corollary C.2) (see Subsubsection 1.4.2.2).

Let K be a field, Q a finite connected quiver and I an admissible ideal of the path algebra KQ. We consider right modules, and read paths from right to left. Each morphism between two indecomposable projective modules P_i and P_j is given by a linear combination of paths in Q that goes from i to j.

Definition 1.2.1 *A bound quiver (Q, I) is* gentle *if the following conditions are satisfied:*

(1) Each vertex of Q is the source of at most two arrows and the target of at most two arrows,

(2) For each arrow α of Q, there is at most one arrow β such that $\beta\alpha$ is a path and $\beta\alpha \notin I$ and at most one arrow γ such that $\alpha\gamma$ is a path and $\alpha\gamma \notin I$,

(3) For each arrow α of Q, there is at most one arrow δ such that $\delta\alpha$ is a path and $\delta\alpha \in I$ and at most one arrow ε such that $\alpha\varepsilon$ is a path and $\alpha\varepsilon \in I$,

(4) The ideal I is generated by paths of length 2.

A finite dimensional algebra Λ is gentle *if it is isomorphic to an algebra of the form KQ/I with (Q, I) a gentle bound quiver.*

A possibly infinite dimensional algebra of the form KQ/I for a quiver Q and an ideal I of KQ satisfying the above definition is called a *locally gentle algebra*.

Indecomposable objects in the bounded derived category of a gentle algebra were classified in terms of homotopy strings and bands in [BM03]. Our notations are based on [ALP16](Section 2) and [OPS25](Section 2.1).

Let (Q, I) be a gentle bound quiver, and let s and t be the source and target maps of Q. For each arrow $\alpha \in Q_1$ we introduce its formal inverse $\overline{\alpha}$, given by $s(\overline{\alpha}) = t(\alpha)$ and $t(\overline{\alpha}) = s(\alpha)$. The set of formal inverses is denoted by $\overline{Q_1}$ and we view the operation $a \mapsto \overline{a}$ has an involution on $Q_1 \sqcup \overline{Q_1}$ by setting $\overline{\overline{a}} = a$. Note that for a concatenation of paths $\rho_2\rho_1$ in Q, the formal inverse is $\overline{\rho_2\rho_1} = \overline{\rho_1}\,\overline{\rho_2}$.

Definition 1.2.2

(1) A direct string *is either a trivial path or a path $a = a_n \dots a_1$ with each a_i in Q_1, and such that $a \notin I$.*

(2) An inverse string *is a path $b = \overline{b_1} \dots \overline{b_n}$ with each $\overline{b_i}$ in $\overline{Q_1}$, and such that $\overline{b} \notin I$.*

(3) A finite (reduced) homotopy string *is a concatenation $\sigma = \sigma_r \dots \sigma_1$ with $t(\sigma_i) = s(\sigma_{i+1})$ for all $1 \leq i \leq r-1$, where:*

(a) *Each* σ_i *is a direct or inverse string,*

(b) *If* σ_i *and* σ_{i+1} *are both direct, then* $\sigma_{i+1}\sigma_i \in I$*,*

(c) *If* σ_i *and* σ_{i+1} *are both inverse, then* $\overline{\sigma_{i+1}\sigma_i} \in I$*,*

(d) *For* $1 \leq i \leq r-1$*,* $\sigma_i \neq \overline{\sigma_{i+1}}$*.*

(4) *A* homotopy band *is a finite homotopy string* $\sigma = \sigma_r \ldots \sigma_1$ *with* $t(\sigma_i) = s(\sigma_{i+1})$ *for* $1 \leq i \leq r-1$*, such that:*

(a) σ *has an equal number of direct and inverse strings* σ_i*,*

(b) $t(\sigma_r) = s(\sigma_1)$ *and* $\sigma_r \neq \overline{\sigma_1}$*,*

(c) $\sigma \neq \tau^m$ *for some homotopy string* τ*, and* $m \geq 2$*.*

(5) *A* left infinite homotopy string *is an infinite concatenation* $\sigma = \ldots \sigma_2\sigma_1$ *satisfying:*

(a) *For all* $k \geq 1$*,* $\sigma_k \ldots \sigma_1$ *is a homotopy string,*

(b) *There exists* $j \geq 1$ *such that* $\ldots \sigma_{j+1}\sigma_j$ *is periodic and for all* $i \geq j$*,* σ_i *is an inverse string of length one.*

(6) *A* right infinite homotopy string $\sigma = \sigma_{-1}\sigma_{-2}\ldots$ *is the inverse* $\overline{\tau}$ *of some left infinite homotopy string* τ*.*

(7) *A* two-sided infinite homotopy string *is a sequence* $\ldots \sigma_1\sigma_0\sigma_{-1}\ldots$ *where* $\ldots \sigma_1\sigma_0$ *(resp.* $\sigma_0\sigma_{-1}\ldots$*) is a left (resp. right) infinite homotopy string.*

For simplicity, we call here a finite homotopy string what is called a finite *reduced* homotopy string in [OPS25]. This property corresponds to the assumption $\sigma_i \neq \overline{\sigma_{i+1}}$. An *infinite* homotopy string will refer to a right, left, or two-sided homotopy string, and a homotopy string will refer to a finite or infinite homotopy string.

Definition 1.2.3

(1) *A* graded finite homotopy string (σ, μ) *is a finite homotopy string* $\sigma = \sigma_r \ldots \sigma_1$ *together with a sequence of integers* $\mu = (\mu_r, \ldots, \mu_0)$ *satisfying for* $i \neq r$*:*

$$\mu_{i+1} = \begin{cases} \mu_i + 1 \text{ if } \sigma_{i+1} \text{ is a direct string,} \\ \mu_i - 1 \text{ if } \sigma_{i+1} \text{ is an inverse string.} \end{cases} \tag{1.1}$$

(2) *A* graded infinite homotopy string (σ, μ) *is an infinite string* σ *together with a sequence of integers* $\mu = (\mu_i)$ *satisfying Equation 1.1 for all* i*.*

(3) *A* graded homotopy band (σ, μ) *is a homotopy band* $\sigma = \sigma_r \ldots \sigma_1$ *together with a sequence of integers* $\mu = (\mu_r, \ldots, \mu_0)$ *satisfying Equation 1.1 for* $i \neq r$*. Note that the definition of a homotopy band ensures that* $\mu_r = \mu_0$*.*

The sequence of integers μ is called a grading*. The shift $\mu[1]$ of a grading μ is the grading defined by $\mu[1]_i = \mu_i - 1$.*

The choice of a grading on σ is determined by the choice of an integer μ_0. One can associate to each graded homotopy string or band a complex in the bounded derived category of the gentle algebra as follows. We use the cohomological convention for complexes.

Definition 1.2.4 [BM03](Section 4)

(1) Let (σ, μ) be a graded homotopy string. The string object *associated to (σ, μ) is the complex $P^\bullet_{(\sigma,\mu)}$ of $D^b(mod\ \Lambda)$ defined by:*

(a) For $k \in \mathbb{Z}$,

$$P^k_{(\sigma,\mu)} = \bigoplus_{\mu_i = k} P_{v_i},$$

where v_i is the target of σ_i or the source of σ_{i+1}, whichever is well-defined,

(b) The differential has components $\sigma_i : P_{v_{i-1}} \to P_{v_i}$ when σ_i is direct and $\overline{\sigma_i} : P_{v_i} \to P_{v_{i-1}}$ when σ_i is inverse.

(2) Let (σ, μ) be a graded homotopy band with $\sigma = \sigma_r \dots \sigma_1$, let M be an indecomposable $K[X]$-module of finite dimension $\dim_K M = m$, and let J be the matrix of the multiplication by X in a chosen basis. The band object *associated to (σ, μ, J) is the complex $P^\bullet_{(\sigma,\mu,J)}$ of $D^b(mod\ \Lambda)$ defined by:*

(a) For $k \in \mathbb{Z}$,

$$P^k_{(\sigma,\mu,J)} = \bigoplus_{\substack{\mu_i = k \\ i \neq r}} P^{\oplus m}_{v_i},$$

where v_i is defined as before,

(b) For $i \neq r$, the differential has a component $\sigma_i I_m : P^{\oplus m}_{v_{i-1}} \to P^{\oplus m}_{v_i}$ when σ_i is direct and a component $\overline{\sigma_i} I_m : P^{\oplus m}_{v_i} \to P^{\oplus m}_{v_{i-1}}$ when σ_i is inverse, where I_m is the identity matrix,

(c) For $i = r$, σ_r induces a component $\sigma_r J : P^{\oplus m}_{v_{r-1}} \to P^{\oplus m}_{v_0}$ when it is direct and a component $\overline{\sigma_r} J : P^{\oplus m}_{v_0} \to P^{\oplus m}_{v_{r-1}}$ when it is inverse.

When K is algebraically closed, one can choose a basis of M such that J is the Jordan block $J_m(\lambda)$ of size m, associated to the scalar $\lambda \in K$. The elements m and λ are called the *parameters* of the band object $P^\bullet_{(\sigma,\mu,J_m(\lambda))}$.

Let $\mathbf{ind}(D^b(mod\text{-}\Lambda))$ be a set of representatives for the isomorphism classes of indecomposable objects in the bounded derived category $D^b(mod\text{-}\Lambda)$ of a gentle algebra $\Lambda = KQ/I$.

Let $\mathbf{St}$ be a set of representatives for the graded homotopy strings of (Q, I) under the equivalence relation identifying a string with its inverse and let $\mathbf{Ba}$ be a set of representatives for the graded homotopy bands of (Q, I) under the equivalence relation identifying a band with its cyclic rotations and their inverses.

The following is a reformulation in our notations of the classification theorem of [BM03].

Theorem 1.2.5 [BM03](Theorem 3) *Suppose that K is algebraically closed. There is a bijection:*

$$\mathbf{ind}(D^b(mod\,\Lambda)) \overset{1-1}{\longleftrightarrow} \mathbf{St} \sqcup (\mathbf{Ba} \times K^* \times \mathbb{N}^*)$$

associating a graded homotopy string or band to the object defined in Definition 1.2.4.

1.3 Reminder on A_∞-categories

We recall the definition of an A_∞-category and its category of twisted complexes, and present a notion of A_∞-localizations and homotopy transfer. This exposition is based on [Sei08, HKK17, AP24].

Definition 1.3.1 *A $\mathbb{Z}$-*graded category without multiplications *$\mathcal{A}$ is given by a class of objects $Ob(\mathcal{A})$, and by a $\mathbb{Z}$-graded K-vector space $\mathcal{A}(X,Y)$ for each pair of objects X,Y in $Ob(\mathcal{A})$, called morphism space. The set of homogeneous morphisms are denoted $\mathcal{A}^k(X,Y)$ for $k\in\mathbb{Z}$, and the degree of an element a of this set is denoted $|a|=k$. The reduced degree of a is $\|a\|=|a|-1$.*

An A_∞-category *is the data of a $\mathbb{Z}$-graded category without multiplications $\mathcal{A}$, together with linear maps:*

$$\mu^n : \mathcal{A}(X_{n-1},X_n)\otimes\ldots\otimes\mathcal{A}(X_0,X_1)\to\mathcal{A}(X_0,X_n)$$

of degree $2-n$, for each integer $n\geq 1$ and collection $X_0,\ldots,X_n$ of objects of $\mathcal{A}$. Moreover, these maps should satisfy the A_∞-relations*:*

$$\sum_{\substack{0\leq k,\ 1\leq j\\ k+j\leq n}} (-1)^{\|a_k\|+\ldots+\|a_1\|}\mu^{n-j+1}(a_n,\ldots,a_{k+j+1},\mu^j(a_{j+k},\ldots,a_{k+1}),a_k,\ldots,a_1)=0$$

for all homogeneous morphisms $a_1,\ldots,a_n$.

The μ^n are called the higher multiplications of $\mathcal{A}$, and we say that the $\mathbb{Z}$-graded category without multiplications $\mathcal{A}$ is endowed with the A_∞-structure μ^n. When several A_∞-category are under consideration, we write $\mu^n_{\mathcal{A}}$ to specify the underlying $\mathbb{Z}$-graded category without multiplications.

A morphism $e_Y\in\mathcal{A}^0(Y,Y)$ is a unit *if:*

- *For all X,Z in $\mathcal{A}$ and homogeneous morphisms $a\in\mathcal{A}(X,Y)$ and $b\in\mathcal{A}(Y,Z)$,*

$$\mu^2(a,e_Y)=a \text{ and } \mu^2(e_Y,b)=(-1)^{|b|}b,$$

- *For $n\neq 2$, $\mu^n(\ldots,e_Y,\ldots)=0$.*

The A_∞-category $\mathcal{A}$ is said to be strictly unital *if each object admits a unit.*

Units are necessarily unique. Note that the A_∞-relation for $n=1$ says that μ^1 is a differential on each $\mathcal{A}(X,Y)$. Let $H^*\mathcal{A}(X,Y)$ be the cohomology of $\mathcal{A}(X,Y)$, seen as a complex with differential μ^1.

Definition 1.3.2 *The* cohomological category *of a strictly unital A_∞-category $\mathcal{A}$ is the $\mathbb{Z}$-graded category $H^*\mathcal{A}$ whose objects are given by $Ob(H^*\mathcal{A})=Ob(\mathcal{A})$ and whose morphism spaces are the cohomology spaces $H^*\mathcal{A}(X,Y)$. The composition is given by:*

$$[a_2][a_1]:=(-1)^{|a_1|}\mu^2(a_2,a_1).$$

Let $H^0\mathcal{A}$ be the category obtained by keeping only the cohomology spaces in degree zero.

Note that the associativity of the composition is ensured by the A_∞-relation for $n = 3$, since μ^1 is zero in homology.

Definition 1.3.3 *An A_∞-functor $\mathcal{F}$ between two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ is the data of a map $\mathcal{F}$ from $Ob(\mathcal{A})$ to $Ob(\mathcal{B})$, and of linear maps:*

$$\mathcal{F}^n : \mathcal{A}(X_{n-1}, X_n) \otimes \ldots \otimes \mathcal{A}(X_0, X_1) \to \mathcal{B}(\mathcal{F}(X_0), \mathcal{F}(X_n))$$

of degree $1-n$, for each integer $n \geq 1$ and collection of objects $X_0, \ldots, X_n$ of $\mathcal{A}$. These maps must satisfy the following relations:

$$\sum_{1\leq r} \sum_{\substack{1\leq s_1,\ldots,s_r \\ s_1+\ldots+s_r=n}} \mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_n, \ldots, a_{n-s_r+1}), \ldots, \mathcal{F}^{s_1}(a_{s_1}, \ldots, a_1))$$
$$= \sum_{\substack{0\leq k,\ 1\leq j \\ k+j\leq n}} (-1)^{\|a_k\|+\ldots+\|a_1\|} \mathcal{F}^{n-j+1}(a_n, \ldots, a_{k+j+1}, \mu^j_{\mathcal{A}}(a_{k+j}, \ldots, a_{k+1}), a_k, \ldots, a_1),$$

for all homogeneous morphisms $a_1, \ldots, a_n$. The composition of two A_∞-functors $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ is given by:

$$(\mathcal{G} \circ \mathcal{F})^n(a_n, \ldots, a_1) = \sum_{1\leq r} \sum_{\substack{1\leq s_1,\ldots,s_r \\ s_1+\ldots+s_r=n}} \mathcal{G}^r(\mathcal{F}^{s_r}(a_n, \ldots, a_{n-s_r+1}), \ldots, \mathcal{F}^{s_1}(a_{s_1}, \ldots, a_1)).$$

If $\mathcal{A}$ and $\mathcal{B}$ are strictly unital, $\mathcal{F}$ is said to be strictly unital *if for all units e_X of X in $Ob(\mathcal{A})$, one has $\mathcal{F}^1(e_X) = e_{\mathcal{F}(X)}$ and $\mathcal{F}^n(\ldots, e_X, \ldots) = 0$ for $n \geq 2$.*
An A_∞-functor $\mathcal{F}$ satisfying $\mathcal{F}^n = 0$ for $n > 1$ is said to be strict*.*

A strictly unital A_∞-functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ induces a K-linear graded functor $H^*\mathcal{F} : H^*\mathcal{A} \to H^*\mathcal{B}$ defined by $H^*\mathcal{F}([a]) := [\mathcal{F}^1(a)]$, which restricts to a functor $H^0\mathcal{F} : H^0\mathcal{A} \to H^0\mathcal{B}$.

Definition 1.3.4 *A strictly unital functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a* quasi-equivalence *of A_∞-categories if the induced functor $H^*\mathcal{F}$ is an equivalence of categories.*

An A_∞-category $\mathcal{A}$ is *formal* if it is quasi-equivalent to $H^*\mathcal{A}$ (seen as an A_∞-category).

Given an A_∞-category $\mathcal{A}$, one can construct its A_∞-category of twisted complexes $Tw\mathcal{A}$. The first step is to pass to the additive enlargement.

Definition 1.3.5 *Let $\mathcal{A}$ be an A_∞-category. The* additive enlargement *of $\mathcal{A}$ is the A_∞-category $add\mathbb{Z}\mathcal{A}$ whose:*

- *Objects are given by formal direct sums $\bigoplus_{X\in Ob(\mathcal{A})} V_X \otimes X$ where V_X is a finite dimensional graded vector space and where only finitely many V_X are non-zero.*
- *Morphism spaces are given by:*

$$add\mathbb{Z}\mathcal{A}(V_X \otimes X, V_Y \otimes Y) := Hom_K(V_X, V_Y) \otimes_K \mathcal{A}(X, Y),$$

and its extension to direct sums by additivity.

- *Higher multiplications are given by:*

$$\mu^n_{add\mathbb{Z}\mathcal{A}}(\varphi_n \otimes a_n, \ldots, \varphi_1 \otimes a_1) = (-1)^{\sum\limits_{1\leq i<j\leq n} |\varphi_i|\|a_j\|} \varphi_n \circ \ldots \circ \varphi_1 \otimes \mu^n_{\mathcal{A}}(a_n, \ldots, a_1). \tag{1.2}$$

for all homogeneous morphisms $\varphi_1 \otimes a_1, \ldots, \varphi_n \otimes a_n$*.*

Notations 1.3.6 *For* $d \in \mathbb{Z}$*, let* $K[d]$ *be the graded vector space with only* K *in degree* d*. For* X *in* $\mathcal{A}$*, let* $X[d] := K[d] \otimes X$ *in* $add\mathbb{Z}\mathcal{A}$*. By letting:*

$$\left(\bigoplus_{X\in Ob(\mathcal{A})} V_X \otimes X\right) \oplus \left(\bigoplus_{X\in Ob(\mathcal{A})} W_X \otimes X\right) := \bigoplus_{X\in Ob(\mathcal{A})} (V_X \oplus W_X) \otimes X,$$

every object of $add\mathbb{Z}\mathcal{A}$ *can be seen as a finite direct sum of objects of the form* $X[d]$*. Moreover, morphisms in* $add\mathbb{Z}\mathcal{A}(X[d], Y[c])$ *are of the form* $\lambda s^{c-d} \otimes a$*, where* $\lambda \in K$*,* a *is in* $\mathcal{A}(X, Y)$*, and* $s^d \in Hom^{-d}(K, K[d])$ *is induced by the identity. We usually write* $\mathcal{A}(X[d], Y[c])$ *instead of* $add\mathbb{Z}\mathcal{A}(X[d], Y[c])$*.*

Definition 1.3.7 *Let* $\mathcal{A}$ *be an* A_∞*-category. The* A_∞*-category* $Tw\mathcal{A}$ *of* twisted complexes *over* $\mathcal{A}$ *is given by:*

- *Objects are pairs* (W, δ)*, called* twisted complexes*, where* W *is an object of* $add\mathbb{Z}\mathcal{A}$ *and* δ *is a morphism in* $add\mathbb{Z}\mathcal{A}^1(W, W)$*, such that:*
  - W *admits a decomposition* $W = \bigoplus_{1\leq i\leq r} W_i$ *for which* δ *is strictly upper triangular,*
  - *The (finite by the previous assumption) sum* $\sum\limits_{n\in\mathbb{N}^*} \mu^n_{add\mathbb{Z}\mathcal{A}}(\delta, \ldots, \delta)$ *is zero.*
- *Morphism spaces are given by:*

$$Tw\mathcal{A}((W, \delta), (Z, \varepsilon)) := add\mathbb{Z}\mathcal{A}(W, Z).$$

- *Higher multiplications are given by:*

$$\mu^n_{Tw\mathcal{A}}(a_n, \ldots, a_1) = \sum_{0\leq k_1,\ldots,k_n} \mu^{n+k_1+\ldots+k_n}_{add\mathbb{Z}\mathcal{A}}(\delta_n, \ldots, \delta_n, a_n, \delta_{n-1}, \ldots, \delta_{n-1}, a_{n-1}, \ldots, a_1, \delta_0, \ldots, \delta_0), \tag{1.3}$$

where each a_i *is in* $Tw\mathcal{A}((W_{i-1}, \delta_{i-1}), (W_i, \delta_i))$*, and each sequence* $(\delta_i, \ldots, \delta_i)$ *between* a_i *and* a_{i+1} *is of length* $k_i \in \mathbb{Z}_{\geq 0}$*. Once again, choosing the* δ_i *to be strictly upper triangular shows that this sum is finite.*

The morphism δ is called the *differential* of the twisted complex (W, δ). Sending the objects of $\mathcal{A}$ to twisted complexes concentrated in degree zero defines an embedding of $\mathcal{A}$ into $Tw\mathcal{A}$.

When $\mathcal{A}$ is strictly unital, so is $Tw\mathcal{A}$ with identity matrices as units. A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ which is strictly unital induces a strictly unital functor $Tw\mathcal{F} : Tw\mathcal{A} \to Tw\mathcal{B}$. See [Sei08](Subsection (3m)) for the details of this construction. Moreover, we have the following proposition:

Proposition 1.3.8 [Sei08](Lemma 3.25) *If $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then the induced functor $Tw\mathcal{F} : Tw\mathcal{A} \to Tw\mathcal{B}$ is also a quasi-equivalence.*

An A_∞-functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ for which the induced functor $Tw\mathcal{F} : Tw\mathcal{A} \to Tw\mathcal{B}$ is a quasi-equivalence is called a *Morita equivalence*. The construction $Tw\mathcal{A}$ is justified by the following proposition:

Proposition 1.3.9 [Sei08](Proposition 3.29) *Let $\mathcal{A}$ be a strictly unital A_∞-category. The cohomological category $H^0 Tw\mathcal{A}$ admits a structure of triangulated category.*

The translation functor is $H^0 S$ where $S : Tw\mathcal{A} \to Tw\mathcal{A}$ is the strict A_∞-functor given by:

$$\big(\sum_i X_i[d_i], \delta\big) \mapsto \big(\sum_i X_i[d_i + 1], S(\delta)\big),$$

where S applied on morphisms is given by $S(\varphi \otimes a) = (-1)^{|\varphi|}\varphi \otimes a$ and its extension by additivity. The distinguished triangles are the ones isomorphic to a diagram:

$$W_0 \xrightarrow{[c]} W_1 \xrightarrow{[i]} Cone(c) \xrightarrow{[p]} S(W_0),$$

where

$$Cone(c) = \left(S(W_0) \oplus W_1, \begin{pmatrix} S(\delta_{W_0}) & 0 \\ -S(c) & \delta_{W_1} \end{pmatrix}\right),$$

and $i = \begin{pmatrix} 0 \\ e_{W_1} \end{pmatrix}$, $p = \begin{pmatrix} S(e_{W_0}) & 0 \end{pmatrix}$, with e_{W_k} a representative of the identity of W_k in homology. The definition of S gives an isomorphism:

$$Tw\mathcal{A}^0(S(W_0), S(W_1)) \simeq Tw\mathcal{A}^1(S(W_0), W_1),$$

ensuring that $S(c)$ as the good domain and codomain.

The category $H^0 Tw\mathcal{A}$ will be denoted by $\mathcal{A}^{tr}$. For a strictly unital functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, let $\mathcal{F}^{tr}$ be the induced functor between $\mathcal{A}^{tr}$ and $\mathcal{B}^{tr}$.

Proposition 1.3.10 [Sei08](Lemma 3.30) *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a strictly unital functor. The induced functor $\mathcal{F}^{tr}$ is triangulated.*

Thus a quasi-equivalence $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ induces an triangulated equivalence $\mathcal{F}^{tr}$ between $\mathcal{A}^{tr}$ and $\mathcal{B}^{tr}$. Moreover, by [Sei08](Lemma 3.28 and 3.33), the inclusion $Tw\mathcal{A} \to Tw(Tw\mathcal{A})$ is a quasi-equivalence. Any full subcategory $\mathcal{B}$ of $Tw\mathcal{A}$ that contains $\mathcal{A}$ induces an equivalence $\mathcal{A}^{tr} \simeq \mathcal{B}^{tr}$.

1.3.1 Localization and homotopy transfer

Given a strictly unital A_∞-category $\mathcal{A}$ and a full-subcategory $\mathcal{B}$, the A_∞-localization $\mathcal{D}(\mathcal{A}|\mathcal{B})$ was introduced in [LO06]. We present here the definition given in [HKK17] (Subsection 3.5), for the special case where $\mathcal{B}$ is a full subcategory of $\mathcal{A}$ with one object.

Definition 1.3.11 *Let $\mathcal{A}$ be an A_∞-category and $\mathcal{B}$ a full subcategory of $\mathcal{A}$ with one object B. The* localization *of $\mathcal{A}$ at $\mathcal{B}$ is the A_∞-category $\mathcal{D} := \mathcal{D}(\mathcal{A}|\mathcal{B})$ given by:*

- *Objects in $\mathcal{D}$ are the same as those of $\mathcal{A}$,*
- *Morphism spaces are given by a decomposition as vector spaces $\mathcal{D}(X,Y) = \bigoplus_{n\in\mathbb{N}^*} \mathcal{D}^{\langle n\rangle}(X,Y)$, where $\mathcal{D}^{\langle 1\rangle}(X,Y) = \mathcal{A}(X,Y)$ and*

$$\mathcal{D}^{\langle n\rangle}(X,Y) = \mathcal{A}(B,Y)\otimes K[1]\otimes(\mathcal{A}(B,B)\otimes K[1])^{\otimes n-2}\otimes\mathcal{A}(X,B)$$

  for $n \geq 2$. The elements of $\mathcal{D}^{\langle n\rangle}(X,Y)$ are linear combinations of terms of the form $a_n \cdot \ldots \cdot a_1$, where $a_1 \in \mathcal{A}(X,B)$, $a_n \in \mathcal{A}(B,Y)$, and the other a_i are in $\mathcal{D}(B,B)$.

  The degrees in $\mathcal{D}$ are given by:

$$|a_n \cdot \ldots \cdot a_1| = |a_n| + \ldots + |a_1| - (n-1).$$

  Thus, $\|a_n \cdot \ldots \cdot a_1\| = \|a_n\| + \ldots + \|a_1\|$.

- *For $r \geq 1$ and $0 = n_0 < n_1 < \cdots < n_r$, higher multiplications are given by:*

$$\mu^r_{\mathcal{D}}(a_{n_r}\cdot\ldots\cdot a_{n_{r-1}+1},\ldots,a_{n_1}\cdot\ldots\cdot a_1) = \sum_{\substack{j\geq 0\\ 1\leq k\leq n_1\\ n_r\geq k+j\geq n_{r-1}+1}} (-1)^{\|a_{k-1}\cdot\ldots\cdot a_1\|} a_{n_r}\cdot\ldots\cdot a_{k+j+1}\cdot\mu^{j+1}_{\mathcal{A}}(a_{k+j},\ldots,a_k)\cdot a_{k-1}\cdot\ldots\cdot a_1 \tag{1.4}$$

Note that B becomes a zero object in $H^*\mathcal{D}$ since $\mu^1_{\mathcal{D}}(e_B\cdot e_B) = e_B$. Moreover when $\mathcal{A}$ is strictly unital, so is $\mathcal{D}(\mathcal{A}|\mathcal{B})$.

There are several notions of A_∞-quotients. See the introduction of [BLM08] for a presentation and for relevant references. We will see in the next subsection that when $\mathcal{A}$ is a differential graded (DG) category, the A_∞-quotient of [LO06] coincides with the quotient of DG categories introduced by Drinfeld [Dri04].

In [Kad80], Kadeishvili showed that any DG algebra admits a quasi-equivalent A_∞-structure on its homology, and in fact the construction holds for an arbitrary A_∞-category. More precisely, for any A_∞-category $\mathcal{A}$, there is an A_∞-structure $\mu_{H^*\mathcal{A}}$ on the $\mathbb{Z}$-graded category without multiplications underlying the cohomological category $H^*\mathcal{A}$, and a quasi-equivalence of A_∞-categories from $H^*\mathcal{A}$ endowed with $\mu_{H^*\mathcal{A}}$ to $\mathcal{A}$. We will present here a notion of transfer of A_∞-structure following [Sei08] (Subsection (1i)). See Remark 1.15 of this reference for more background and references on this notion.

Let $\mathcal{A}$ be an A_∞-category such that each complex $(\mathcal{A}(X,Y),\mu^1_{\mathcal{A}})$ is split into a complex $E(X,Y)$ with zero differential and an acyclic complex $D(X,Y)$. Let T^1 be an endomorphism (of graded vector space) of $\mathcal{A}(X,Y)$ of degree -1 which vanishes on $E(X,Y)$ and is a chain homotopy from zero to the identity of $D(X,Y)$, that is:

$$\tilde{\mu}^1_{\mathcal{A}}T^1 + T^1\tilde{\mu}^1_{\mathcal{A}} = \mathcal{F}^1\mathcal{G}^1 - \mathrm{id},$$

where $\mathcal{F}^1$ is the inclusion of $E(X,Y)$ in $\mathcal{A}(X,Y)$ and $\mathcal{G}^1$ the projection onto $E(X,Y)$ with respect to this decomposition.

Proposition 1.3.12 [Sei08](Proposition 1.12, Remark 1.13)
Let $\mathcal{C}$ be the $\mathbb{Z}$-graded category without multiplications whose objects are $Ob(\mathcal{C}) = Ob(\mathcal{A})$, and morphism spaces are $\mathcal{C}(X,Y) = E(X,Y)$ for all X,Y in $Ob(\mathcal{A})$. The following recursive formulas give an A_∞-structure on $\mathcal{C}$, as well as a quasi-equivalence $\mathcal{F} : \mathcal{C} \to \mathcal{A}$:

$$\mu^n_{\mathcal{C}}(a_n, \dots, a_1) = \sum_{2\leq r} \sum_{\substack{1\leq s_1,\dots,s_r \\ s_1+\dots+s_r=n}} T^1(\mu^r_{\mathcal{A}}(\mathcal{F}^{s_r}(a_n, \dots, a_{n-s_r+1}), \dots, \mathcal{F}^{s_1}(a_{s_1}, \dots, a_1))),$$
$$\mathcal{F}^n(a_n, \dots, a_1) = \sum_{2\leq r} \sum_{\substack{1\leq s_1,\dots,s_r \\ s_1+\dots+s_r=n}} \mathcal{G}^1(\mu^r_{\mathcal{A}}(\mathcal{F}^{s_r}(a_n, \dots, a_{n-s_r+1}), \dots, \mathcal{F}^{s_1}(a_{s_1}, \dots, a_1))).$$

When $E(X,Y) = H^*\mathcal{A}(X,Y)$ for all X and Y, an A_∞-category $\mathcal{C}$ obtained in this way will be called a *minimal model* for $\mathcal{A}$.

1.3.2 Differential graded categories

Up to some signs, differential graded (DG) categories are a special case of A_∞-categories. Our main references are [Dri04, BK91, CC21].

Definition 1.3.13 *A DG category $\mathcal{A}$ is a K-category where each morphism space $\mathcal{A}(X,Y)$ is a complex of K-vector spaces:*

$$\mathcal{A}(X,Y) = (\bigoplus_{k\in\mathbb{Z}} \mathcal{A}(X,Y)^k, d_{\mathcal{A}}),$$

and such that the composition $\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z)$ is a chain map:

$$|g \circ f| = |g| + |f| \text{ and } d_{\mathcal{A}}(g \circ f) = d_{\mathcal{A}}(g) \circ f + (-1)^{|g|} g \circ d_{\mathcal{A}}(f),$$

for f and g homogeneous, and where $|g| = k$ for $g \in \mathcal{A}(Y,Z)^k$.
A DG functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between two DG categories $\mathcal{A}$ and $\mathcal{B}$ is a K-linear functor such that:

$$\mathcal{F} : \mathcal{A}(X,Y) \to \mathcal{B}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a chain map, that is $|\mathcal{F}(f)| = |f|$ and $\mathcal{F}(d_{\mathcal{A}}(f)) = d_{\mathcal{B}}(\mathcal{F}(f))$.

One passes from an A_∞-category $\mathcal{A}$ with $\mu^n = 0$ for $n \geq 3$ to a DG category, and the other way around, by setting:

$$d_{\mathcal{A}}(f) = (-1)^{|f|}\mu^1_{\mathcal{A}}(f) \text{ and } m_{\mathcal{A}}(g,f) := g \circ_{\mathcal{A}} f = (-1)^{|f|}\mu^2_{\mathcal{A}}(g,f). \quad (1.5)$$

The cohomological category of a DG category and the notion of quasi-equivalences between DG categories are defined as before.

The category $\mathcal{A}^{pre-tr}$ of (one sided) twisted complexes of a DG category $\mathcal{A}$ was introduced in [BK91]. We use here the notations of [Dri04](Subsection 2.4).

Its objects are defined as formal finite direct sums $(\bigoplus_i X_i[d_i], \delta)$, where each X_i is in $\mathcal{A}$ and d_i in $\mathbb{Z}$, together with a strictly upper triangular differential $\delta = (\delta_{ji})$ with $\delta_{ji} \in \mathcal{A}(X_i, X_j)[d_j - d_i]$ homogeneous of degree 1, satisfying $d_{\text{naive}}(\delta) + \delta^2 = 0$, where $d_{\text{naive}}(\delta)$ is obtained by applying the (shifted) differential on each component: $d_{\text{naive}}(\delta) = (d_{\mathcal{A}[d_j - d_i]}(\delta_{ji}))$.

The DG structure on the morphism spaces is given by the following rule:

$$d_{\mathcal{A}^{pre-tr}}(f) = d_{\text{naive}}(f) + \eta \circ f - (-1)^{|f|} f \circ \delta, \tag{1.6}$$

where $f = (f_{ji})$, with each f_{ji} in $\mathcal{A}(X_i, Y_j)[c_j - d_i]$, is a homogeneous morphism from $(\bigoplus_i X_i[d_i], \delta)$ to $(\bigoplus_j Y_j[c_j], \eta)$.

One can check that this DG category coincides with the A_∞-category $Tw\mathcal{A}$. For an A_∞-category $\mathcal{A}$ with $\mu^n_{\mathcal{A}} = 0$ for all $n \geq 3$, Equations 1.3 and 1.2 give the following formula for the differential applied on a homogeneous element $s^{c-d} \otimes a$ of $\mathcal{A}(X[d], Y[c])$:

$$\mu^1_{Tw\mathcal{A}}(s^{c-d} \otimes a) = \mu^1_{\text{add}\mathbb{Z}\mathcal{A}}(s^{c-d} \otimes a) + \mu^2_{\text{add}\mathbb{Z}\mathcal{A}}(\eta, s^{c-d} \otimes a) + \mu^2_{\text{add}\mathbb{Z}\mathcal{A}}(s^{c-d} \otimes a, \delta).$$

Passing from an A_∞-category to a DG category using Equation 1.5, the above equality becomes:

$$d_{Tw\mathcal{A}}(s^{c-d} \otimes a) = d_{add\mathbb{Z}\mathcal{A}}(s^{c-d} \otimes a) + m_{Tw\mathcal{A}}(\eta, s^{c-d} \otimes a) - (-1)^{|s^{c-d} \otimes a|} m_{Tw\mathcal{A}}(s^{c-d} \otimes a, \delta). \tag{1.7}$$

The morphim spaces in $Tw\mathcal{A}$ and $\mathcal{A}^{pre-tr}$ are identified by viewing a morphism $s^{c-d} \otimes a$ of the space $\mathcal{A}(X[d], Y[c])$ as a morphism a of $\mathcal{A}(X, Y)[c-d]$, and the other way around. Since:

$$\begin{aligned} d_{add\mathbb{Z}\mathcal{A}}(s^{c-d} \otimes a) &= (-1)^{|s^{c-d} \otimes a|} \mu^1_{add\mathbb{Z}\mathcal{A}}(s^{c-d} \otimes a) \\ &= (-1)^{d-c+|a|} s^{c-d} \otimes \mu^1_{\mathcal{A}}(a) = (-1)^{d-c} s^{c-d} \otimes d_{\mathcal{A}}(a), \end{aligned}$$

and $d_{\mathcal{A}[c-d]}(a) = (-1)^{c-d} d_{\mathcal{A}}(a)$, Equation 1.6 and 1.7 coincide.

We recall the localization of DG categories introduced in [Dri04].

Definition 1.3.14 [Dri04](§3.1) *Let $\mathcal{B}$ be a full subcategory of a DG category $\mathcal{A}$.*
The DG quotient of $\mathcal{A}$ by $\mathcal{B}$ *is the DG category $\mathcal{A}/\mathcal{B}$ given by:*

- *Objects in $\mathcal{A}/\mathcal{B}$ are the same as those of $\mathcal{A}$,*
- *Morphism spaces are given by a decomposition into a direct sum of vector spaces (but not of complexes) $\mathcal{A}/\mathcal{B}(X,Y) = \bigoplus_{n \in \mathbb{N}^*} \mathcal{A}/\mathcal{B}^{\langle n \rangle}(X,Y)$ where:*

$$\mathcal{A}/\mathcal{B}^{\langle n \rangle}(X,Y) = \bigoplus_{(X_i) \in Ob(\mathcal{B})^n} \mathcal{A}(X_{n-1}, X_n) \otimes K[1] \otimes \mathcal{A}(X_{n-2}, X_{n-1}) \otimes \ldots \otimes K[1] \otimes \mathcal{A}(X_0, X_1),$$

  with $X_0 = X$, $X_n = Y$, and the sum is over all collections $(X_i)_{1 \leq i \leq n}$ of objects of $\mathcal{B}$. The elements of $\mathcal{A}/\mathcal{B}^{\langle n \rangle}(X,Y)$ are linear combinaitions of elements of the form $a_n \varepsilon_{n-1} a_{n-1} \ldots a_2 \varepsilon_1 a_1$, where ε_i is the canonical generator of $K[1]$,

- *The differential is given by $d_{\mathcal{A}/\mathcal{B}}(\varepsilon_i) = id_{X_i}$ together with the graded Leibniz rule. See Equation 1.8 for an illustration on a generic element.*

As mentioned before, when $\mathcal{A}$ is viewed as an A_∞-category, Drinfeld's localization coincides with the localization of A_∞-categories of [LO06]. In our case, one can see that for a full subcategory supported on one object, Definition 1.3.14 coincides with Definition 1.3.11.

Let $\mathcal{A}$ be a A_∞-category with $\mu^n_{\mathcal{A}} = 0$ for $n \geq 0$, let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$ with one object B, and let $\mathcal{D} = \mathcal{D}(\mathcal{A}|\mathcal{B})$ be the A_∞-localization. Equation 1.4 shows that $\mu^n_{\mathcal{D}} = 0$ for $n \geq 3$. Let $\mathcal{A}/\mathcal{B}$ be the quotient of $\mathcal{A}$ by $\mathcal{B}$ viewed as DG categories. The morphism spaces of $\mathcal{D}$ and $\mathcal{A}/\mathcal{B}$ are identified by viewing a morphism $a_n \cdot \ldots \cdot a_1$ of $\mathcal{D}(X,Y)$ as a morphism $a_n\varepsilon \ldots \varepsilon a_1$ of $\mathcal{A}/\mathcal{B}(X,Y)$, where $\varepsilon = -\varepsilon_B$.

On one side,

$$\begin{aligned} & d_{\mathcal{A}/\mathcal{B}}(a_n\varepsilon \ldots \varepsilon a_1) \\ &= \sum_{k=1}^{n} (-1)^{|a_n \ldots a_{k+1}\varepsilon|} a_n\varepsilon \ldots d_{\mathcal{A}}(a_k) \ldots \varepsilon a_1 + \sum_{i=1}^{n-1} (-1)^{|a_n \ldots a_{i+1}|} a_n\varepsilon \ldots d_{\mathcal{A}/\mathcal{B}}(\varepsilon) a_i \ldots a_1 \\ &= \sum_{k=1}^{n} (-1)^{|a_n \ldots a_{k+1}\varepsilon|} a_n\varepsilon \ldots d_{\mathcal{A}}(a_k) \ldots \varepsilon a_1 - \sum_{i=1}^{n-1} (-1)^{|a_n \ldots a_{i+1}|} a_n\varepsilon \ldots m_{\mathcal{A}}(a_{i+1}, a_i) \ldots a_1. \end{aligned} \tag{1.8}$$

On the other side, by letting $\diamondsuit_{j-1} = \|a_{j-1} \cdot \ldots \cdot a_1\|$,

$$\begin{aligned} & \mu^1_{\mathcal{D}}(a_n \cdot \ldots \cdot a_1) \\ &= \sum_{k=1}^{n} (-1)^{\diamondsuit_{k-1}} a_n \cdot \ldots \cdot \mu^1_{\mathcal{A}}(a_k) \cdot \ldots \cdot a_1 + \sum_{i=1}^{n-1} (-1)^{\diamondsuit_{i-1}} a_n \cdot \ldots \cdot \mu^2_{\mathcal{A}}(a_{i+1}, a_i) \cdot \ldots \cdot a_1. \end{aligned}$$

The two constructions coincide since $(-1)^{|a_n \cdot \ldots \cdot a_1| + \diamondsuit_{j-1}} = (-1)^{|a_n\varepsilon \ldots \varepsilon a_j|}$.

Since we are working over a field, the DG quotient satisfies the following property:

Theorem 1.3.15 [Dri04](Theorem 3.4) *Let $\mathcal{B}$ be a full subcategory of a DG category $\mathcal{A}$. There is a triangulated equivalence:*

$$(\mathcal{A}/\mathcal{B})^{tr} \simeq \mathcal{A}^{tr}/\mathcal{B}^{tr}.$$

where the right hand side denotes the Verdier localization for triangulated categories.

Example 1.3.16 *Let Λ be a K-algebra and let $\mathcal{C} = proj\text{-}\Lambda$ be the category of projective Λ-modules, seen as a DG category concentrated in degree zero. For a twisted complex $X = (\bigoplus_i X_i[d_i], \delta)$ in $\mathcal{C}^{pre-tr}$, the component δ_{ji} of the differential must be of degree one in $\mathcal{C}(X_i, X_j)[d_j - d_i]$. Thus it can be non zero only if $d_i = d_j + 1$. This shows that X is actually a complex of projective Λ-modules, where $X_i[d_i]$ is in homological degree d_i. The formula of $d_{\mathcal{C}^{pre-tr}}$ shows that morphisms in $H^0(\mathcal{C}^{pre-tr})$ are morphisms of complexes up to homotopy, and thus $\mathcal{C}^{tr}$ coincides with the category of perfect complexes $K^b(proj\text{-}\Lambda)$. The following theorem can be useful for the study of triangulated quotients of this category. See [Sei08](Lemma 3.32) for a similar statement for A_∞-categories.*

Theorem 1.3.17 [BK91](§4 - Theorem 1) *Let $\mathcal{A}$ be a DG category, and let $\mathcal{B}$ be a full subcategory of $\mathcal{A}^{pre-tr}$ supported on objects $X_1, \ldots, X_n$. There is a triangulated equivalence:*

$$\mathcal{B}^{tr} \simeq \langle X_1, \ldots, X_n \rangle,$$

where $\langle X_1, \ldots, X_n \rangle$ is the smallest strictly full triangulated subcategory of $\mathcal{A}^{tr}$ that contains the X_i.

Let $X_1, \dots, X_n$ be a collection of objects of $K^b(proj\text{-}\Lambda)$. Taking $\mathcal{A}$ to be the full subcategory of $\mathcal{C}^{pre-tr}$ supported on $Ob(\mathcal{C})$ and on the X_i, and $\mathcal{B}$ the full subcategory of $\mathcal{A}$ supported on the X_i, one gets the following equivalences:

$$K^b(proj\text{-}\Lambda)/\langle X_1, \dots, X_n\rangle \simeq \mathcal{A}^{tr}/\mathcal{B}^{tr} \simeq (\mathcal{A}/\mathcal{B})^{tr}.$$

1.4 Topological Fukaya categories, after [HKK17]

1.4.1 From Fukaya categories of graded marked surfaces to gentle algebras

1.4.1.1 Graded marked surfaces with arc systems

We recall the definition and basic properties of the topological Fukaya category of a graded marked surface as introduced in [HKK17]. This exposition is based on Sections 2.1 and 3 of this article.

Definition 1.4.1 *A* graded surface *(S, η) is the data of a smooth oriented surface S and a section η of the projectivized tangent bundle $\mathbb{P}(TS)$, called a* grading*.*

A morphism of graded surfaces *from (S_1, η_1) to (S_2, η_2) is the data of an orientation preserving local diffeomorphism $f : S_1 \to S_2$ together with the homotopy class of an homotopy $\tilde{f}$ from $f^*\eta_2$ to η_1. The composition of two morphisms of graded surfaces is:*

$$(f, \tilde{f}) \circ (g, \tilde{g}) = (f \circ g, \tilde{g}(g^*\tilde{f})),$$

where $\tilde{g}(g^*\tilde{f})$ is the concatenation.

A graded curve *$(I, \gamma, \tilde{\gamma})$ in a graded surface (S, η) is the data of an immersion $\gamma : I \to S$ of a one dimensional manifold I, together with an homotopy $\tilde{\gamma}$ in $\Gamma(I, \gamma^*\mathbb{P}(TS))$ from the grading $\gamma^*\eta$ to the tangent space $\dot{\gamma}$.*

Given two (possibly equal) graded curves $(I_1, \gamma_1, \tilde{\gamma}_1)$ and $(I_2, \gamma_2, \tilde{\gamma}_2)$ intersecting transversely at a point $\gamma_1(t_1) = \gamma_2(t_2) =: p$ with $\dot{\gamma}_1(t_1) \neq \dot{\gamma}_2(t_2)$, their intersection index *at p is defined to be the following homotopy class in $\pi_1(\mathbb{P}(T_pS)) \simeq \mathbb{Z}$:*

$$i_p(\gamma_1, \gamma_2) := \tilde{\gamma}_1(t_1) \cdot \kappa \cdot \tilde{\gamma}_2(t_2)^{-1},$$

where for $i = 1$ and 2, $\tilde{\gamma}_i(t_i)$ is the homotopy class of a path from $\eta(p)$ to $\dot{\gamma}_i(t_i)$ induced by the grading γ_i, and where κ is defined by the counterclockwise rotation in T_pS from $\dot{\gamma}_1(t_1)$ to $\dot{\gamma}_2(t_2)$ by an angle strictly smaller than π.

The counterclockwise rotation is the one given by the orientation of S. Note that there is an equality $i_p(\gamma_1, \gamma_2) + i_p(\gamma_2, \gamma_1) = \tilde{\gamma}_1(t_1) \cdot \kappa \cdot \bar{\kappa} \cdot \tilde{\gamma}_1(t_1)^{-1} = 1$, where $\bar{\kappa}$ is induced by the counterclockwise rotation strictly smaller than π from $\dot{\gamma}_2(t_2)$ to $\dot{\gamma}_1(t_1)$.

The *shift* of a graded curve $(I, \gamma, \tilde{\gamma})$ is the graded curve $(I, \gamma[1], \tilde{\gamma}[1])$ where $\gamma[1] = \gamma$ and the homotopy $\tilde{\gamma}[1]$ is obtained by pre-composing at each $p \in S$ the homotopy $\tilde{\gamma}$, restricted to $\mathbb{P}(T_pS)$, by the generator τ of $\pi_1(\mathbb{P}(T_pS))$ induced by the counterclockwise rotation by an angle of π. The intersection index for shifted curves is given by the following equation:

$$i_p(\gamma_1[m], \gamma_2[n]) = \tau^m \cdot \tilde{\gamma}_1(t_1) \cdot \kappa \cdot \tilde{\gamma}_2(t_2)^{-1} \cdot \tau^{-n} = i_p(\gamma_1, \gamma_2) + m - n. \quad (1.9)$$

Shifting a graded curve amounts to post-composing by the shift automorphism $(id_S, [1])$ of (S, η), which is defined in the same way.

One can resolve a crossing with intersection index one in the following way. Let γ, ρ be two graded curves intersecting transversely at $\gamma(x) = \rho(y) = p$ with intersection index $i_p(\gamma, \rho) = 1$. We place ourselves in a disk D around p. Suppose that the domain I of γ is an oriented segment from a to b decomposing as $I = I_1 \cup I_2$, with I_1 from a to x and I_2 from x to b. Orient the domain J of ρ such that it is a segment from c to d with induced cyclic ordering (a, c, b, d), and decompose it similarly as $J_1 \cup J_2$. Let γ_i (resp. ρ_i) be the restriction of γ to I_i (resp. of ρ to J_i). Since $\tilde{\gamma}(x) \cdot \kappa = \tau \cdot \tilde{\rho}(y)$, one can define an immersion δ whose image is homotopic to $\gamma_1 \cdot \rho_2$, together with a grading $\tilde{\delta}$ induced by $\tilde{\gamma}$ on I_1 and $\tilde{\rho}[1]$ on J_2, and by inserting a linear homotopy from $\tilde{\gamma}(x)$ to $\tilde{\gamma}(x) \cdot \kappa$. The same construction gives a grading on $\rho_1 \cdot \gamma_2$.

Definition 1.4.2 *A* marked surface (S, M) *is a smooth oriented surface S with boundary, together with a set $M \subset \partial S$ which is a disjoint union of closed segments or circles, called* marked components*, and such that each compact connected component of ∂S contains finitely many, and at least one, connected component of M.*

An arc *on a marked surface (S, M) is a locally embedded closed interval in S which intersects transversely M at its endpoints, and which is not isotopic to an interval in M by an* isotopy of arc*, that is by an isotopy keeping its endpoints in M. An arc which is isotopic to the closure of a connected component of $\partial S \backslash M$ is called a* boundary arc*.*

An arc system *on (S, M) is a collection of pairwise disjoint non-isotopic arcs. An arc system that includes all boundary arcs and cuts S into polygons is called a* full *arc system.*

A formal *arc system is an arc system that cuts S into polygons, each containing at least one connected component of $\partial S \backslash M$ in its boundary. A* full formal *arc system is a formal arc system that cuts S into polygons, each containing exactly one connected component of $\partial S \backslash M$ in its boundary.*

A morphism of marked surfaces $f : (S_1, M_1) \to (S_2, M_2)$ *is an orientation preserving immersion that satisfies the inclusion $f(M_1) \subset M_2$ and that sends boundary arcs of S_1 to disjoint non-isotopic arcs in S_2.*

If S is compact each connected component of ∂S is either a circle entirely in M or a sequence of segments alternatingly belonging to M and its complement. We will not consider a disk with $\partial S = M$ as a marked surface.

Unless S is a disk with two marked components, note that a full arc system cannot be formal, and thus a full formal arc system is not a full arc system.

The definition of a morphism of marked surfaces f from (S_1, M_1) to (S_2, M_2) is such that any full arc system A_2 on S_2 that includes all the images of the boundary arcs of S_1 under f can be lifted to a full arc system A_1 on S_1. However, the composition of two morphisms of marked surfaces is not a morphism of marked surfaces in general, since boundary arcs may be sent to isotopic arcs. Another direct consequence of the definition is that the image of an unmarked component can not be included in a marked component.

Remark 1.4.3 *A marked surface can also refer to a couple (S, M) where M is a set of distinguished points in the boundary or interior of S. See for instance [OPS25]. More precisely, one defines M as:*

- *$M = M^\circ \sqcup M^\bullet \sqcup P^\circ \sqcup P^\bullet$ is a finite set of points on S. The elements of M°, called $\circ$-points, and of $M^\bullet$, called $\bullet$-points, are points on the boundary ∂S. Elements of $M = M^\circ \sqcup M^\bullet$ are called* marked points*. Each connected component of the boundary is required to contain at least one marked point of each color, and $\circ$-points and $\bullet$-points must alternate on each connected component. The elements of $P = P^\circ \sqcup P^\bullet$ are called respectively $\circ$ and $\bullet$-punctures, and lie on the interior of S.*

In this definition, $\circ$-points are in bijection with the segments of M as in Definition 1.4.2, and $\bullet$-points are in bijection with unmarked boundary segments. A fully marked component as in Definition 1.4.2 corresponds to a $\circ$-puncture. Unmarked boundary components, which we will consider in Remark 1.4.8, correspond to $\bullet$-punctures. Thus with this convention, S can have empty boundary as long as P is not empty.

An arc on (S, M) is called a $\circ$-arc. One defines the dual notion of $\bullet$-arc by requiring that the endpoints are in $P^\bullet$. A full formal arc system is called an admissible dissection *and is usually denoted Δ. One can define the dual notion of an admissible $\bullet$-dissection. Each admissible dissection Δ induces a unique dual admissible $\bullet$-dissection Δ^* such that each $\ell^* \in \Delta^*$ intersects exactly one arc ℓ of Δ.*

A graded admissible dissection (Δ, G) on a marked surface (S, M) is the extra data of an integer $g(\alpha)$ for each minimal angle α of the dissection Δ.

The tuple (S, M, Δ, G) is called a graded marked surface*. We will see in Remark 1.4.7 and Subsubsection 1.4.2.1 how one can pass from this definition to the one given by 1.4.1 and 1.4.2, and the other way around. We will use both conventions depending on the context.*

1.4.1.2 The topological Fukaya category of a graded marked surface

We now give the definition of the minimal A_∞-category of an arc system following Subsection 3.3 of [HKK17]. This category was introduced by Bocklandt in the case where all boundary components are fully marked [Boc16]. In an appendix of this article, Abouzaid showed that it computes the partially wrapped Fukaya category of the surface as defined by Abouzaid-Seidel in [AS10]. According to [LP20], the proof extends to the general case.

Definition 1.4.4 *Let (S, M, η) be a graded marked surface together with a system of graded arcs A. The* minimal A_∞-category of A*, denoted by $\mathcal{F}_A(S)$, is defined by:*

- *The set of objects is the set of arcs A,*
- *A basis for the morphism space $\mathcal{F}_A(S)(\rho, \gamma)$, from a graded arc ρ to a graded arc γ, is given by:*
  - *(a) One identity morphism e_ρ when $\gamma = \rho$,*
  - *(b) Non-constant paths in M which follow the reverse orientation of the boundary, and go from an endpoint of ρ and to an endpoint of γ. These paths are called* boundary paths *and are considered up to reparametrization. The degree of a boundary path a from an intersection p to an intersection q is defined by:*

$$|a| = i_p(\rho, a) - i_q(\gamma, a),$$

where a is given an arbitrary grading.

- *Higher multiplications are given by:*
  - *(a)* $\mu^1 = 0$,
  - *(b) The composition of morphisms is given, up to a sign, by concatenation. For boundary paths a from ρ to γ and b from γ to δ, $\mu^2(b,a) = (-1)^{|a|} ba$ if a and b are composable and $\mu^2(b,a) = 0$ otherwise,*
  - *(c) The non-zero μ^n for $n \geq 3$ are induced by disk sequences. Let (S_0, M_0) be a closed disk with a number $n \geq 3$ of marked components, and let $a_1, \dots, a_n$ be the distinct boundary paths between consecutive boundary arcs, taken in clockwise order. A* disk sequence *in (S, M) is the image of the a_i under a morphism $(S_0, M_0) \to (S, M)$ which sends boundary arcs of (S_0, M_0) to arcs in A. For a disk sequence $(a_n, \dots, a_1)$,*

$$\mu^n(a_n, \dots, a_1 b) = (-1)^{|b|} b \text{ and } \mu^n(c a_n, \dots, a_1) = c,$$

for each basis morphism b and c such that $a_1 b \neq 0$ and $c a_n \neq 0$.

For a boundary path a, its degree $|a|$ as a morphism of $\mathcal{F}_A(S)$ is independent of the choice of a grading for a, as it can be seen using the Equation 1.9 for shifted intersection indices:

$$|a[1]| = i_p(\rho, a[1]) - i_q(\gamma, a[1]) = i_p(\rho, a) - 1 - i_q(\gamma, a) + 1.$$

For a disk sequence $(a_n, \dots, a_1)$, $\sum_i |a_i| = n - 2$ and μ^n has the desired degree. The fact that there is no ambiguity in applying μ^n for $n \geq 3$ comes from the following argument. Let $(a_n, \dots, a_1)$ be a disk sequence and let ρ_i be the arc on which a_i stops. Then $\rho_n a_n \dots \rho_1 a_1$ is a null-homotopic loop.

- Let $(b_n, \dots, b_1)$ be a sequence of composable morphisms, and let $b_1 = ba = b'a'$ be two decompositions such that $(b_n, \dots, b_2, b)$ and $(b_n, \dots, b_2, b')$ are both disk sequences. Since $\rho_n b_n \dots \rho_1 b$ and $\rho_n b_n \dots \rho_1 b'$ are null-homotopic loops, both a and a' are homotopic to $\rho_n b_n \dots \rho_1 b_1$ by homotopies fixing their endpoints. Thus a and a' are equal as morphism.

- Let $(b_n, \dots, b_1)$ be a sequence of composable morphisms, and let $b_1 = ba$ and $b_n = cb'$ be two decompositions such that $(b_n, \dots, b_2, b)$ and $(b', b_{n-1}, \dots, b_1)$ are both disk sequences. As before, let ρ_i be the arc on which b_i stops, and let ρ_0 be the arc on which b_1 starts. Then $\rho_n b_n \dots b_2 \rho_1 b$ and $b' \rho_{n-1} b_{n-1} \dots b_1 \rho_0$ are null-homotopic loops, and:

$$a\rho_0 \simeq \rho_n c b' \rho_{n-1} b_{n-1} \dots b_2 \rho_1 b a \rho_0 \simeq \rho_n c.$$

  Thus ρ_0 is isotopic to ρ_n. It forces the endpoint and starting point of a to coincide, and similarly for c. But boundary paths arising from a disk sequence cannot loop several times around a boundary component, otherwise following the boundary arcs and paths around the immersed disk would give a non-contractible curve. This shows that both a and c are identity morphisms.

The same argument shows that the dual statement holds, and ensures that a decomposition involving a disk sequence is unique.

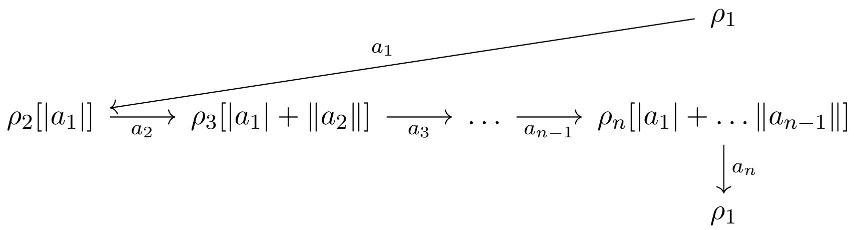
It is proved in Proposition 3.1 of [HKK17] that the higher multiplications μ^n defined in this way satisfy the A_∞-relations. The minimality of this A_∞-category refers to the fact that μ^1 is zero.

Let (S_1, M_1, η_1) and (S_2, M_2, η_2) be graded marked surfaces, with systems of graded arcs A_i on S_i. A morphism of graded marked surfaces f from S_1 to S_2 that sends graded arcs in A_1 to graded arcs in A_2 induces a strict A_∞-functor f_* from $\mathcal{F}_{A_1}(S_1)$ to $\mathcal{F}_{A_2}(S_2)$. The only relations that need to be satisfied come from disk sequences. They hold since an immersed marked disk in S_2, with boundary paths in the image of f, lifts to an immersed marked disk in S_1 with boundary arcs in A_1.

The following proposition shows that for full arc systems, the A_∞-category $\mathcal{F}_A(S)$ does not depend on the choice of A.

Proposition 1.4.5 [HKK17](Proposition 3.2) *Let (S, M, η) be a graded marked surface, and let A and B be two full systems of graded arcs on S. The minimal A_∞-categories $\mathcal{F}_A(S)$ and $\mathcal{F}_B(S)$ are Morita equivalent.*

The proof is based on the following observation. It shows that the higher multiplications of $\mathcal{F}_A(S)$ were chosen to reflect the topology of the underlying surface. Let S be a graded marked surface and A a system of graded arcs on S. Let $(a_n, \ldots, a_1)$ be a disk sequence in $\mathcal{F}_A(S)$, and let ρ_i be the arc on which a_i stops. Consider the following composition of morphisms of twisted complexes:



For $a_i : \rho_i[e] \to \rho_{i+1}[d]$, $|a_i|_{Tw\mathcal{F}_A(S)} = |s^{d-e} \otimes a_i|_{Tw\mathcal{F}_A(S)} = |a_i| + e - d$. Equation 1.3 and 1.2 give:

$$\begin{aligned}\mu^2_{Tw\mathcal{F}_A(S)}(a_n, a_1) &= \mu^2_{add\mathbb{Z}\mathcal{F}_A(S)}(a_n, a_1) + \mu^n_{add\mathbb{Z}\mathcal{F}_A(S)}(a_n, \ldots, a_1) \\ &= \mu^n_{add\mathbb{Z}\mathcal{F}_A(S)}(s^{|a_n|} \otimes a_n, s^{\|a_{n-1}\|} \otimes a_{n-1}, \ldots, s^{\|a_2\|} \otimes a_2, s^{|a_1|} \otimes a_1) \\ &= (-1)^{\diamondsuit} s^0 \otimes \mu^n_{\mathcal{F}_A(S)}(a_n, \ldots, a_1) = (-1)^{\diamondsuit} e_{\rho_1},\end{aligned}$$

where $\diamondsuit = \sum_{j=3}^{n} \|a_j\| (\sum_{i=2}^{j-1} \|a_i\|) + (\sum_{j=2}^{n} \|a_j\|)|a_1| = \sum_{j=3}^{n} \|a_j\| (\sum_{i=2}^{j-1} \|a_i\|) - (|a_1| + 1)|a_1|$.

Thus if B is obtained from A by adding a graded arc ρ_1 that bounds an immersed marked disk giving rise to such a disk sequence, ρ_1 is isomorphic to a twisted complex supported on arcs in A. The inclusion functor $i_* : \mathcal{F}_A(S) \to \mathcal{F}_B(S)$ is therefore a quasi-equivalence. Moreover, shifting the grading of an arc gives a Morita equivalence.

It is recalled in the proof that for any two full systems of graded arcs, one can pass from one to the other by a finite sequence of these operations and their inverses. In fact, any two such sequences will give isomorphic equivalences after passing to homology. This is because one has a functor $A \mapsto \mathcal{F}_A(S)$ from the category whose objects are full arc systems up to isotopy on (S, M) and whose morphisms are inclusions, to the category of strictly unital A_∞-categories and Morita equivalences. Then one uses a theorem of Harer which says that the classifying space of full arc systems is contractible [Har85, Har86]. In this sense the equivalences are canonical, and this allows one to define *the topological Fukaya category* of a graded marked surface S, denoted $\mathcal{F}(S)$, as any of the Morita equivalent categories $Tw\mathcal{F}_A(S)$, for a full arc system A on S. The triangulated topological Fukaya category $H^0\mathcal{F}(S)$ will be denoted $\mathcal{W}(S)$.

Note that passing from a full arc system B to a full formal arc system A, by removing boundary arcs, gives Morita equivalent A_∞-categories. In general, one can find a full formal arc system by first finding a full arc system that cuts S into a single polygon. This is how the following lemma is proved:

Proposition 1.4.6 [HKK17](Lemma 3.3) *Every compact connected graded marked surface which has at least one boundary arc admits a full formal arc system.*

Remark 1.4.7 *A graded quiver Q together with a homogeneous ideal of relations I in KQ gives rise to a $\mathbb{Z}$-graded category $\mathcal{P}(Q,I)$ whose objects are vertices and whose morphism spaces are given by $\mathcal{P}(Q,I)(u,v) = e_v(KQ/I)e_u$.*

For a full formal arc system A on S, the minimal A_∞-category $\mathcal{F}_A(S)$ is a $\mathbb{Z}$-graded category of this form, for a quiver Q obtained in the following way. The vertices of Q are given by the arcs in A, and the set of arrows is the set of boundary paths which start and end at arcs of A, but does not cross any other arcs. The ideal I is generated by quadratic relations coming from composable arrows which do not correspond to composable paths. By construction, the quiver with relations (Q,I) is gentle in the sense of Definition 1.2.1.

Thus the image of A under a quasi-equivalence $F : Tw\mathcal{F}_A(S) \to Tw\mathcal{F}_B(S)$ gives a formal generator of $Tw\mathcal{F}_B(S)$, whose cohomological endomorphism A_∞-algebra is isomorphic to the graded gentle algebra KQ/I. Here, the term generating is to be taken in the sense that the inclusion $F(A) \hookrightarrow Tw\mathcal{F}_B(S)$ induces a quasi-equivalence from $Tw(F(A))$ to $Tw(Tw\mathcal{F}_B(S))$. See section (3j) of [Sei08] for a definition of a generator of a triangulated A_∞-category.

Remark 1.4.8 *A gentle quiver with relations (Q,I) induced by a full formal arc system A cannot contain a cycle of arrows $a_1,\ldots,a_n$ with quadratic relations at each vertex. Otherwise this cycle would correspond to a polygon cut out by A which does not contain a connected component of $\partial S\backslash M$.*

Gentle algebras with such a cycle nonetheless appear as endomorphism algebra of formal generators, when considering marked surfaces with unmarked boundary components. Given a graded marked surface with unmarked boundary components (S,M), one can define its topological Fukaya category as a subcategory of $\mathcal{F}(S,M')$, where M' is obtained from M by adding a marked component on each unmarked boundary component. See [HKK17](Lemma 5.1) for details.

1.4.2 From gentle algebras to Fukaya categories

Remark 1.4.7 illustrated how gentle algebras arise naturally when studying the topological Fukaya category of a graded marked surface. This connection between gentle algebras and surfaces with triangulations, or dissections, was first made in [Lab09, ABCP10], where the authors were inspired by the theory of cluster algebras from triangulated surfaces [FG09, FST08]. In works such as [OPS25, PPP19, LP20, BCS21, BCS23], a graded marked surface with admissible dissection was associated to any graded gentle algebra. Some aspects of the representation theory of the algebra were then translated in terms of curve combinatorics on the surface. These dictionaries are referred to as *geometric models* for the gentle algebra.

1.4.2.1 Marked surfaces with admissible graded dissections from gentle bound quivers

Let $\Lambda = KQ/I$ be a graded gentle algebra. We recall the construction of [LP20, OPS25], associating to Λ a marked surface with graded admissible dissection (S, Δ). To simplify this exposition, we suppose that Q is acyclic. The cyclic case is similar to the acyclic one, the difference being that each oriented cycle of Q (taken up to rotation) will give rise to a $\circ$-puncture on S.

A non trivial path $p = \alpha_{n-1} \dots \alpha_2\alpha_1 \in KQ$ is called permitted if the α_i's are distinct and for $i = 1, \dots, n-2$, one has $\alpha_{i+1}\alpha_i \notin I$. It is a *permitted thread* if it is maximal, in the sense that for all $\beta \in Q_1$ neither βp not $p\beta$ is a permitted path. A trivial path e_v associated to $v \in Q_0$ is a permitted thread if one of the following conditions hold:

- v is the target of exactly one arrow and the source of none,
- v is the source of exactly one arrow and the target of none,
- v is the target of exactly one arrow γ and the source of exactly one arrow β, satisfying $\beta\gamma \notin I$.

Note that the introduction of the trivial permitted threads ensures that each vertex $v \in Q_0$ belongs to exactly two permitted threads. Technically in order to include the case $Q = A_1$, one should add twice the permitted thread e_v, where v is the only vertex.

The first step is to define a graph R whose vertices are in bijection with the permitted threads of (Q, I), and whose edges are in bijection with the vertices of Q, each vertex of Q connecting the two permitted threads he belongs to. The set of edges incident to a vertex p of R is equipped with a total order by following the order in which the corresponding vertices of Q appear in the permitted thread p. By definition, this extra data turn R into what is called a *marked ribbon graph*.

Now R encodes all the data needed to build (S, Δ). To each vertex $v \in R$ of valency d, let P_v be an oriented $2d$-polygon, whose sides are numbered $1, \dots, 2d$ by following the boundary according to the orientation. Let $(e_{d-1}, \dots, e_0)$ be the totally ordered set of edges incident to v, and label each odd side $2k+1$ of P_v by e_k. Each edge e of R, joining vertices v and w, is the label of a side s_v of P_v and s_w of P_w. The surface S is obtained by identifying for all edges e of R the sides s_v and s_w. For each vertex v of R, a $\circ$-point is placed on the side numbered $2d$ of the polygon P_v. The dissection Δ is obtained by joining in each polygon P_v, the $\circ$-point to the center of each odd side $2k+1$, by non-intersecting lines. Each of these lines will contribute to half of a $\circ$-arc of Δ. The $\bullet$-points are placed in such a way that they alternate with the $\circ$-points on the boundary, and each boundary component containing no $\circ$-point is replaced by a $\bullet$-punctures. Each cycle (taken up to rotation) $\alpha_n \dots \alpha_2\alpha_1 \in KQ$ with the α_i's distinct and $\alpha_{i+1}\alpha_i \in I$ for $i \in \mathbb{Z}/n\mathbb{Z}$, gives rise to such a $\bullet$-puncture.

An alternative way to obtain the same surface with admissible dissection is given in [PPP19] (Definition 4.6), by associating a lozenge to each arrow of a blossoming quiver $Q^{❀}$, and giving instructions on how to glue them.

This construction and the one of Remark 1.4.7 are inverses of each other, and give the following theorem which was proven in different forms in [BCS21, OPS25, LP20, PPP19], and stated explicitly as Theorem 4.10 in [PPP19].

Theorem 1.4.9 *There is a bijection between the set of isomorphism classes of gentle bound quivers and the set of homeomorphism classes of marked surfaces with admissible dissection.*

By construction, the arrows of Q are in bijection with the minimal angles of Δ, and thus a grading on Q induces a grading on Δ. We now explain how one can use this grading to construct a line field η on S.

The line field will be transverse to each $\circ$-arc of Δ. Thus it suffice to describe its restriction to the polygons obtained by cutting S along Δ. It is more convenient to work here with the convention that the markings on ∂S are given by marked segments rather than marked points. Once again to simplify the exposition, we also suppose that S does not have $\bullet$-punctures. In this way, cutting S along Δ gives a collection of polygons, each of which has an even number of sides which are alternatively boundary arcs of S and $\circ$-arcs. Moreover, among the sides which are boundary arcs of S, exactly one is unmarked.

Now we will use the fact that a line field on a polygon is determined by the winding numbers of its sides. A precise definition of the winding number of a graded curve on a graded surface is given for example in [APS23](Definition 3.5). See also [Chi72] for a detailed exposition on line fields induced by vector fields. Intuitively, the winding number of a graded curve γ computes the number of U-turns the line given by the line field η makes relatively to the one given by the grading of γ. The winding number of a boundary arc coincides with its degree given in Definition 1.4.4.

Let P be one of the polygons obtained by cutting S along Δ. Let $2m$ be the number of its sides, and let $a_1, \dots, a_m$ be the sides corresponding to boundary arcs of S, taken in cyclic order, with a_m being the only unmarked boundary arcs. By [LP20](Theorem 1.8 and Equation 2.1), any choice of integers θ_i for $i = 1, \dots, m$, satisfying $\sum_{i=1}^{m} \theta_i = m - 2$ will determine a line field η_P on P such that θ_i coincides with the winding number of a_i. Using the condition $\sum_{i=1}^{m} \theta_i = m - 2$, θ_m is determined by the θ_i for $i = 1, \dots, m-1$. We choose the θ_i for $i = 1, \dots, m-1$ to be the integers given by the grading on Δ. The line field η on S is then taken to be the one whose restriction on each P coincides with η_P.

This construction leads to the following equivalence.

Theorem 1.4.10 *[LP20](Theorem 3.11) Let Λ be a homologically smooth graded gentle algebra, and let (S, M, Δ, G) be the associated marked surface with graded admissible dissection. There is a line field $\eta(\Delta, G)$ on S giving an equivalence:*

$$per(\Lambda) \simeq \mathcal{W}(S, M, \eta(\Delta, G)).$$

1.4.2.2 A geometric model for the bounded derived category of a gentle algebra

New directions for the study of derived equivalence for graded gentle algebras were based on this connection with graded marked surfaces, and have eventually led to the establishment of a complete derived invariant [LP20, APS23, JSW25].

Another aspect of this connection is reflected by the fact that indecomposable objects in the derived category of a graded gentle algebra Λ, have a natural geometric interpretation. In [HKK17] (Theorem 4.3), the authors showed that isomorphism classes of indecomposable objects in the triangulated topological Fukaya category of a graded marked surface S are in bijection with isotopy classes of admissible curves on S, equipped with indecomposable local systems. Based on a different method and working without the homologically smooth hypothesis, [OPS25](Corollary C.2) gives a classification of the indecomposable of $\mathcal{D}_{fd}(\Lambda)$ in terms of (twisted, possibility unbounded) complexes called strings and bands. Here $\mathcal{D}_{fd}(\Lambda)$ denotes the full subcategory of the derived category of Λ, seen as differential graded algebra with zero differential, given by the differential graded modules with finite dimensional total cohomology. Then [OPS25] (Theorem 2.13) gives an explicit bijection between string and band complexes and graded arcs on the corresponding graded marked surface (together with an indecomposable $K[X]$-module in the case of a band complex).

We now give a non-exhaustive list of objects and algebraic constructions in the derived category of Λ, which admit a geometric counterpart on the corresponding graded marked surface.

- In [OPS25](Theorem 3.3), morphisms between indecomposable objects are realized as oriented graded intersections between the corresponding graded curves,
- In [OPS25](Theorem 4.1), the mapping cone of a morphism is realized as the resolution of the corresponding oriented graded intersections,
- Auslander-Reiten triangles can be obtained by rotating the endpoints of a graded curve along the boundary of the surface [OPS25](Theorem 5.1 and Corollary 5.4),
- According to [APS23](Theorem 5.2), the indecomposable summands of a silting objects must give rise to an admissible dissection of the marked surface,
- In [CS23](Theorem 3.4), the mutation of a silting object is realized as the flipping of a diagonal in a quadrilateral.

1.4.3 Localizations of the topological Fukaya category at a collection of arcs

The localization of the topological Fukaya category at a boundary arc was done in [HKK17] using the A_∞-localization of Definition 1.3.11. In [CJS23], the localization at a collection of arcs was expressed in term of recollements, a notion first introduced in [BBD].

1.4.3.1 Recollements for DG quotients

Recollements of triangulated categories are closely related to Verdier and Bousfield localization of triangulated categories. See [Kra10] for a detailed exposition.

Definition 1.4.11 [BBD] *A* recollement of triangulated categories *is a diagram of triangulated functors:*

$$\mathcal{Q} \underset{q^!}{\overset{q^*}{\leftarrow}} \xrightarrow{q_*} \mathcal{T} \underset{i^!}{\overset{i^*}{\leftarrow}} \xrightarrow{i_*} \mathcal{S}$$

where:

- $(q^*, q_*, q^!)$ *and* $(i^*, i_*, i^!)$ *are adjoint triples,*
- q_*, i^* *and* $i^!$ *are fully faithfull,*
- $q^* \circ i^*$, $i_* \circ q_*$ *and* $q^! \circ i^!$ *are zero,*
- *For all* X *in* $\mathcal{T}$ *there are triangles:*

$$i^* i_* X \to X \to q_* q^* X \to i^* i_* X[1],$$
$$q_* q^! X \to X \to i^! i_* X \to q_* q^! X[1].$$

We recall here some results on derived categories of DG categories, following [AL17, CC21, Gye24]. The main reference on this topic is [Kel94].

Let $\mathcal{A}$ be a DG category. The opposite DG category of $\mathcal{A}$ is the DG category $\mathcal{A}^{op}$ which has the same objects as $\mathcal{A}$ and $Hom_{\mathcal{A}^{op}}(a, b) = Hom_{\mathcal{A}}(b, a)$ for $a, b \in \mathcal{A}$. The composition is given by:

$$\beta \circ_{\mathcal{A}^{op}} \alpha = (-1)^{|\alpha||\beta|} \alpha \circ_{\mathcal{A}} \beta.$$

The following differential gives to the category $Mod\text{-}K$ of complexes of K-modules a structure of DG category:

$$d(f) = d_Y \circ f - (-1)^l f \circ d_X,$$

for a homogeneous morphism $f \in Hom^l_K(X, Y) = \bigoplus_{j-i=l} Hom_K(X^i, Y^j)$ between complexes (X, d_X) and (Y, d_Y).

A right DG module M over $\mathcal{A}$ is a DG functor $M : \mathcal{A}^{op} \to Mod\text{-}K$. For $a \in \mathcal{A}$, we write M_a for the complex of K-modules $M(a)$. We treat right $\mathcal{A}^{op}$-modules as left $\mathcal{A}$ modules by writing the fibers ${}_aM$ instead of M_a. For $\mathcal{B}$ a DG category, an $\mathcal{A}$-$\mathcal{B}$ bimodule is a right $\mathcal{A}^{op} \otimes \mathcal{B}$ module, where $\mathcal{A}^{op} \otimes \mathcal{B}$ is the tensor DG category (see for instance Subsubsection 2.1.2 of [AL17] for a definition). Let $\mathcal{A}$-Mod-$\mathcal{B}$ be the DG category of $\mathcal{A}$-$\mathcal{B}$ bimodules.

The diagonal $\mathcal{A}$-$\mathcal{A}$ bimodule, denoted $\mathcal{A}$, is defined by ${}_a\mathcal{A}_b = Hom_{\mathcal{A}}(b, a)$ for a and b in $\mathcal{A}$. The action on morphisms f and g in $\mathcal{A}$ is given by:

$$\mathcal{A}(f \otimes g) = (-1)^{|g||-|} f \circ (-) \circ g.$$

Let $\mathcal{C}$ be another DG category. The Hom functors

$$Hom_{\mathcal{B}}(-,-) : \mathcal{A}\text{-}Mod\text{-}\mathcal{B} \otimes \mathcal{C}\text{-}Mod\text{-}\mathcal{B} \to \mathcal{C}\text{-}Mod\text{-}\mathcal{A},$$
$$Hom_{\mathcal{B}}(-,-) : \mathcal{B}\text{-}Mod\text{-}\mathcal{A} \otimes \mathcal{B}\text{-}Mod\text{-}\mathcal{C} \to \mathcal{A}\text{-}Mod\text{-}\mathcal{C}$$

are defined by:

$${}_cHom_{\mathcal{B}}(M, N)_a = Hom_{\mathcal{B}}({}_aM, {}_cN),$$
$${}_aHom_{\mathcal{B}}(M, N)_c = Hom_{\mathcal{B}}(M_a, N_c).$$

The *tensor* functor

$$(-) \otimes_{\mathcal{B}} (-) : \mathcal{A}\text{-}Mod\text{-}\mathcal{B} \otimes \mathcal{B}\text{-}Mod\text{-}\mathcal{A} \to \mathcal{A}\text{-}Mod\text{-}\mathcal{C}$$

is defined by:

$$_aM \otimes_{\mathcal{B}} N_c = Coker(\varphi : {}_aM \otimes_K \mathcal{B} \otimes_K N_c \to {}_aM \otimes_K N_c),$$

where ${}_aM \otimes_K N_c = \bigoplus_{b \in \mathcal{B}} {}_aM_b \otimes_K {}_bN_c$ is the tensor product of complexes of K-modules, and where ${}_aM \otimes_K \mathcal{B} \otimes_K N_c = \bigoplus_{b,b' \in \mathcal{B}} {}_aM_b \otimes_K {}_b\mathcal{B}_{b'} \otimes_K {}_{b'}N_c$. The morphism φ is given by:

$$\varphi(x \otimes \beta \otimes y) = M(a,\beta)(x) \otimes y - x \otimes N(\beta, c)(y).$$

Given two DG functors $f : \mathcal{A} \to \mathcal{A}'$, $g : \mathcal{B} \to \mathcal{B}'$ and an $\mathcal{A}' \otimes \mathcal{B}'$ bimodule M, one gets an $\mathcal{A} \otimes \mathcal{B}$ bimodule ${}_fM_g$ by *restriction of the scalars along f and g*:

$$_a({}_fM_g)_b = {}_{f(a)}M_{g(b)}.$$

Definition 1.4.12 *Let $f : \mathcal{A} \to \mathcal{B}$ be a DG functor. It induces the following DG functors between the modules categories:*

- *The* extension of scalars
$$f^* : Mod\text{-}\mathcal{A} \to Mod\text{-}\mathcal{B}$$
is the functor defined by $(-) \otimes_{\mathcal{A}} {}_f\mathcal{B}$,

- *The* restriction of scalars
$$f_* : Mod\text{-}\mathcal{B} \to Mod\text{-}\mathcal{A}$$
is the functor defined by $(-) \otimes_{\mathcal{B}} \mathcal{B}_f$. It sends a $\mathcal{B}$ module M to the restriction M_f,

- *The* twisted extension of scalars
$$f^! : Mod\text{-}\mathcal{A} \to Mod\text{-}\mathcal{B}$$
is the functor defined by $Hom_{\mathcal{A}}(\mathcal{B}_f, -)$.

The category $Z^0(Mod\text{-}\mathcal{A})$ admits an exact structure whose conflations are sequences of morphisms that induce split short exact sequences at the level of fibers. It is a Frobenius exact category whose stable category identifies with $H^0(Mod\text{-}\mathcal{A})$, which thus inherits the structure of a triangulated category. See [Kel94](Section 2) for details.

The derived category $D(\mathcal{A})$ of $\mathcal{A}$ is constructed by inverting quasi-isomorphisms in $H^0(Mod\text{-}\mathcal{A})$, that is, morphisms that induce quasi-isomorphisms at the level of fibers. It is done by taking the Verdier quotient $H^0(Mod\text{-}\mathcal{A})/Ac(\mathcal{A})$, where $Ac(\mathcal{A})$ is the full subcategory of acyclic modules, that is, modules whose fibers are acyclic complexes.

The perfect derived category $per(\mathcal{A})$ of $\mathcal{A}$ is the subcategory of compact objects of $D(\mathcal{A})$, that is, of objects M for which $Hom_{D(\mathcal{A})}(M, -)$ commutes with infinite direct sums. The DG category $\mathcal{A}$ is said to be *homologically smooth* if $\mathcal{A}$ is a perfect $\mathcal{A}$-$\mathcal{A}$ bimodule. It is *proper* if the total cohomology of each of its morphism spaces is finitely generated and $D(\mathcal{A})$ is compactly generated.

An $\mathcal{A}$ module P is *DG-projective* if it satisfies the following universal property: for all surjective quasi-isomorphism $X \to Y$ and morphism $P \to Y$, both in $Z^0(Mod\text{-}\mathcal{A})$, there exists a morphism

$P \to X$ also in $Z^0(Mod\text{-}\mathcal{A})$ that makes the diagram commutes. One can show that DG-projective modules are the ones that are projective as graded $\mathcal{A}$-modules and that satisfy

$$Hom_{H^0(Mod\text{-}\mathcal{A})}(P, N) = 0$$

for all acyclic module N. The notion of DG-injective module is defined dually.

Each $\mathcal{A}$-module is quasi-isomorphic to a DG-projective (resp. injectif) module and this yield a triangle equivalence between $D(\mathcal{A})$ and the homotopy category of DG-projective (resp. injective) modules. Derived funtors are defined as usual by passing through these equivalences.

One can use the Tensor-Hom adjunction to check that (f^*, f_*) and $(f_*, f^!)$ form pairs of adjoint functors. The restriction f_* sends acyclic modules to acyclic modules. As a consequence, f^* preserves DG-projective modules and $f^!$ preserves DG-injective modules. This induces the derived functors:

$$Lf^* : D(\mathcal{A}) \to D(\mathcal{B}),\ f_* : D(\mathcal{B}) \to D(\mathcal{A}) \text{ and } Rf^! : D(\mathcal{A}) \to D(\mathcal{B}).$$

We still denote Lf^* by f^* and $Rf^!$ by $f^!$.

Theorem 1.4.13 [Gye24](Th. 3.1), [CC21](Th 5.1.3). *Let $\mathcal{B}$ be a strictly full subcategory of a DG category $\mathcal{A}$. Let $i : \mathcal{B} \to \mathcal{A}$ be the inclusion and $q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the funtor induced by taking the Drinfeld quotient. There is a recollement:*

$$D(\mathcal{A}/\mathcal{B}) \; \begin{array}{c} \xleftarrow{\;q^*\;} \\ \xrightarrow{\;q_*\;} \\ \xleftarrow{\;q^!\;} \end{array} \; D(\mathcal{A}) \; \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow{\;i^!\;} \end{array} \; D(\mathcal{B})$$

where the functors are the one defined in 1.4.12. It induces a triangle equivalence up to direct summands, that is when taking the idempotent closure,

$$per(\mathcal{A})/per(\mathcal{B}) \to per(\mathcal{A}/\mathcal{B}).$$

1.4.3.2 Localization at a collection of arcs

In [CS23], the authors studied silting reductions for perfect derived categories of gentle algebras. They described the orbit category of the reduction, under the shift functor, by cutting the surface along arcs. This was generalized in [CJS23] where the geometric interpretation has been shown to hold without the need to pass to the orbit category. These results were used for example in [JSW25] to study further the silting theory and derived equivalences for graded gentle algebras.

The following gentle algebras will appear as generators of the quotient.

Definition 1.4.14 [CJS23](Definition 2.1) *Let $A = KQ/\langle I \rangle$ be a graded gentle algebra and let e be sum $e_1 + \ldots + e_m$ of vertex indempotents. Let $J \subseteq I$ be the set quadratic monomial relations that go through a vertex in $\{1, \ldots, m\}$. Let $J_1 = Q_1$ and $J_s = \{\alpha_s \ldots \alpha_1 \mid \alpha_{i+1}\alpha_i \in J \text{ for } 1 \leq i < n\}$ for $s \geq 2$.*

The algebra $A_e = KQ_e/\langle I_e \rangle$ is defined as follows:

(1) *The vertices of Q_e is $Q_0 \backslash \{1, \ldots, m\}$,*

(2) *The arrows of Q_e are of the form $[\alpha_s \dots \alpha_1] : s(\alpha_1) \to t(\alpha_s)$ for $\alpha_s \dots \alpha_1 \in J_s$ such that $s(\alpha_1)$ and $t(\alpha_s)$ are not in $\{1, \dots, m\}$,*

(3) $|[\alpha_s \dots \alpha_1]| = \sum_{i=1}^{n} |\alpha_i| - (s-1)$,

(4) $I_e = \{[\beta_t \dots \beta_1][\alpha_s \dots \alpha_1] \mid \beta_1 \alpha_s \in I\}$.

See **??** for an example. The following theorem was proven for a larger class of algebras called graded quadratic monomial algebras.

Theorem 1.4.15 [CJS23](Theorem 2.10 and 4.1) *Let $A = KQ/I$ be a graded gentle algebra and let e be a sum of vertex idempotents. There is a recollement:*

$$D(A_e) \underset{q^!}{\overset{q^*}{\leftarrow}} \xrightarrow{q_*} D(A) \underset{i^!}{\overset{i^*}{\leftarrow}} \xrightarrow{i_*} D(eAe)$$

If two of the three algebras A, eAe and A_e are homologically smooth and proper, then so is the third, and it induces a recollement of triangulated topological Fukaya categories:

$$\mathcal{W}(S_e) \leftarrow \rightarrow \leftarrow \mathcal{W}(S) \leftarrow \rightarrow \leftarrow \mathcal{W}(eSe)$$

where S_e, S and eSe are the graded marked surfaces associated to A_e, A and eAe.

The proof relies on the computation of a partial cofibrant DG algebra resolution for A. When the three algebras are homologically smooth and proper, it is shown in [CJS23] that the first recollement restricts to a recollement of the bounded derived categories. When A is both smooth and proper, its bounded dervied category coincides with its perfect derived category. One then uses the equivalence of Theorem 1.4.10 to state it in terms of triangulated topological Fukaya categories.

One can easily see in its geometric model if a gentle algebra is smooth or proper. A graded gentle algebra is homologically smooth if an only its associated surface has no unmarked boundary component. It is proper if and only if it has no fully marked boundary components [LP20](Lemma 3.3).

The graded marked surfaces of the theorem can be described in the following way. Let (S, M, Δ, G) be the marked surface with graded admissible dissection associated to A, and let L be the collection of graded arcs in Δ that correspond to the idempotent e.

The graded marked surface $(S_e, M_e, \Delta_e, G_e)$ is obtained by successively cutting (S, M, Δ, G) along the arcs of L. Given an arc $\ell \in L$, the marked surface (S_ℓ, M_ℓ) is obtained by cutting S along ℓ, and considering each new boundary segment induced by ℓ as part M_ℓ. The arc collection $\Delta \setminus \ell$ induces an admissible dissection Δ_ℓ on (S_ℓ, M_ℓ). The ungraded gentle algebra associated to (S_ℓ, M_ℓ) coincides with A_ℓ (viewed as an ungraded algebra). Finally, the grading of A_ℓ induces a grading G_ℓ on Δ_ℓ.

The graded marked surface eSe is constructed using the ribbon graph induced by L, and the grading is induced by G. Alternatively, it is obtained by cutting S along $\Delta^* \setminus L^*$, where Δ^* is the dual dissection.

When cutting a surface, we ignore the components corresponding to spheres (without boundary) that have a unique ∘-puncture and a unique •-puncture, as well as disks that have a unique ∘-point and a unique •-point on there boundary.

1.5 Main results

Throughout this section, we assume that the base field is of characteristic zero.

1.5.1 Main results of Chapter 2

In Chapter 2, we study the localization of the derived category $\mathcal{D}(\Lambda)$ of a graded gentle algebra Λ, by a spherical band object. We show that this localization is equivalent to the derived category of an algebra $\Lambda_{(\alpha,\beta)}$ (Theorem 2.1.3). This motivates us to define a class of algebras given by quivers with relations, that we call *pinched gentle algebras* (Definition 2.1.6).

In Definition 2.4.8 and Remark 2.4.9, we introduce the notion of a marked surface with conical singularities, as the topological space obtained by the contraction of a collection of disjoint simple closed curves on a smooth marked surface. We also define the notion of a graded simple admissible dissection on a marked surface with conical singularities and show the following:

Proposition 1.5.1 *(Proposition 2.4.10) Graded pinched gentle quivers are in one to one correspondence with marked surfaces with conical singularities endowed with a graded simple admissible dissection.*

In Definition 2.1.9, we define a processus to obtained a new pinched gentle algebra $\Lambda_{(\alpha,\beta)}$ from a pinched gentle algebra Λ containing a subquiver of a certain type, called an acyclic graded Kronecker and denoted (α,β). Moreover, each acyclic graded Kronecker gives rise to an object in $per(\Lambda)$, called a band object supported on (α,β). The main result is the following:

Theorem 1.5.2 *(Theorems 2.1.11 and 2.1.10) Let Λ be a graded pinched gentle algebra with an acyclic graded Kronecker (α,β). There is a recollement:*

$$\mathcal{D}(\Lambda_{(\alpha,\beta)}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}(\Lambda) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}(K[x]/(x^2)) \;\; ,$$

where x is of degree 1. It induces a triangle equivalence:

$$per(\Lambda)/thick(B) \simeq per(\Lambda_{(\alpha,\beta)}),$$

where B is a band object supported on (α,β).

Let Λ be a graded pinched gentle algebra, and let S_Λ be the associated marked surface with conical singularities endowed with a graded simple admissible dissection. One can apply Theorem 1.5.2 to describe the localization of $per(\Lambda)$ at a simple closed curve γ of winding number zero on S_Λ (not passing through a singularity), in the following way.

By applying Theorem 1.5.2 a finite amount of time, one can show that $per(\Lambda)$ can be realized as a quotient of the perfect derived category of a gentle algebra $\hat{\Lambda}$. Derived equivalences for $\hat{\Lambda}$, which are given by [HKK17](Proposition 3.2), allow us to show Proposition 2.4.3 which states that Λ is derived equivalent to a graded pinched gentle algebra Λ' for which γ corresponds to a band object supported on an acyclic graded Kronecker (α, β). One can then apply Theorem 1.5.2.

The strategy to prove Theorem 1.5.2 is as follows. We first realize $\mathcal{D}(\Lambda)$ as the derived category $\mathcal{D}(\mathcal{A})$ of a DG category $\mathcal{A}$ containing a well chosen strictly full subcategory $\mathcal{B}$, before applying Theorem 1.4.13. In Proposition 2.5.5, we show that the DG quotient $\mathcal{A}/\mathcal{B}$ is formal and we give an explicit description of its cohomological category. For this, we use a spectral sequence for the morphism spaces of a DG quotient, whose first page is described in a general way in Proposition 2.3.4. It is worth noting that the computation of the other pages, carried out in Proposition 2.5.10, are made possible by the fact that the object B is spherical, and thus enjoys a simple homology. Finally, we show in Lemma 2.6.2 that $H^*(\mathcal{A}/\mathcal{B})$ is Morita equivalent to the pinched gentle algebra $\Lambda_{(\alpha,\beta)}$.

1.5.2 Main results of Chapter 3

Chapter 3 essentially generalizes the results obtained in Chapter 2, by enlarging the class of pinched gentle algebras and realizing them as generators for localizations of partially wrapped Fukaya categories of marked surfaces by spherical objects.

The main strategy here differs from the one used in Chapter 2 by the fact that we work this time with an A_∞-enhancement of the triangulated quotient. However, even though the proofs are different, the computation of the homology of a formal generator are the similar (compare the definition of Ψ in the proof of Proposition 2.5.10, and the one of T in Notation 3.5.6).

We first define the class of pinched gentle algebras in Definition 3.3.2, generalizing Definition 2.1.6, and the notion of an admissible dissection on a marked surface with conical singularities, generalizing the simple admissible dissections of Definition 2.4.8.

We then associate in Subsection 3.3.3 a pinched quiver with relations (Q, I) to each graded admissible dissection A of a marked surface with conical singularities S, and we define the category $\mathcal{F}_A(S)$ to be the path category $\mathcal{P}(Q, I)$ whose objects are the vertices in Q_0 and morphism spaces are $\mathcal{P}(Q, I)(i, k) = e_k(KQ/\langle I \rangle)e_i$ (Definition 3.3.6).

Now suppose that S is a marked surface with one conical singularity obtained by contracting a simple closed curve γ of winding number zero, on a smooth graded marked surface $\hat{S}$. Let A be an admissible dissection on S. By Subsection 3.3.4, it lifts naturally to an admissible dissection $\hat{A}$ of $\hat{S}$. Our main theorem is:

Theorem 1.5.3 *(Theorem 3.5.3) There is a Morita equivalence:*

$$\mathcal{F}_A(S) \to \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B}).$$

Here $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$ denotes the A_∞-quotient (see Definition 1.3.11) of the topological Fukaya category $\mathcal{F}(\hat{S})$ of $\hat{S}$ by the full subcategory $\mathcal{B}$ supported on objects which are isomorphic to elements in $thick(B)$ after passing to the zero homology, where B is a spherical band object associated to γ (see Subsection 3.3.5).

As an immediate consequence, one can use [HKK17](Proposition 3.2) which asserts that the Morita equivalence class of $\mathcal{F}_{\hat{A}}(\hat{S})$ is independent of $\hat{A}$, to see that the same holds for $\mathcal{F}_A(S)$ (for choices of graded admissible dissections on S inducing the same grading on $\hat{S}$). Moreover, computations not included in this thesis suggest that Theorem 1.5.3 can be used to classify the indecomposable objects of $\mathcal{F}_A(S)^{tr}$ by using graded curves on S, in analogy with [HKK17](Theorem 4.3). Inspired by [HKK17], we call $\mathcal{F}(S) := Tw\mathcal{F}_A(S)$ the topological Fukaya category of S.

One can iterate Theorem 1.5.3 to obtain a statement involving marked surfaces with several conical singularities.

The proof of Theorem 1.5.3 goes as follows. In Lemma 3.5.2, we show that the admissible dissection A (seen as a subset of $\hat{A}$) generates $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$. We use this to define a full subcategory $\mathcal{A}$ of $\mathcal{F}(\hat{S})$ wich will induced a generator of $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$ by taking an A_∞-quotient $\mathcal{D}(\mathcal{A}|B)$. In order to simplify the description of $\mathcal{D}(\mathcal{A}|B)$ we compute in Proposition 3.4.8 a minimal model for $\mathcal{A}$, denoted $H^*\mathcal{A}$, which induces a new generator $\mathcal{D} := \mathcal{D}(H^*\mathcal{A}|B)$ of $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$. It is shown in Proposition 3.5.10 that $\mathcal{D}$ is formal. As was the case in Chapter 2, the computation of $H^*\mathcal{D}$ is made possible by the fact that B is a spherical object and hence enjoys a simple homology (see the proof of Proposition 3.5.7). Finally we show in Theorem 3.5.11 that the cohomological category of $\mathcal{D}$ can be described as the path category of the pinched quiver associated to A, ie. as $\mathcal{F}_A(S)$. A diagram summarizing the situation is presented in Subsection 3.1.1.

We give two bases for pinched gentle algebras along the way. One in Remark 3.5.14, and one in Proposition 3.5.15 obtained using Bergman's Diamond Lemma.

In the last Section 3.6, we use the (triangulated) topological Fukaya category of a pinched marked surface to give an example of a non-Krull-Schmidt triangulated category containing two silting objects having a different number of indecomposable summands. This illustrates how [AI12](Corollary 2.28) can fail when one drops the Krull-Schmidt assumption.

Chapter 2

Recollements for graded gentle algebras from spherical band objects

2.1 Introduction

In this chapter, we prove that for a gentle algebra Λ, the localization of the derived category $\mathcal{D}(\Lambda)$ by a spherical band object is equivalent to the derived category of an algebra $\Lambda_{(\alpha,\beta)}$ which we describe explicitly by quiver and relations (see Definition 2.1.6). We call $\Lambda_{(\alpha,\beta)}$ a *pinched gentle algebra*. It sits in a recollement (see Theorem 2.1.3):

$$\mathcal{D}(\Lambda_{(\alpha,\beta)}) \leftrightarrows \mathcal{D}(\Lambda) \leftrightarrows \mathcal{D}(K[x]/(x^2)) \ .$$

This localization process can be iterated (see Theorems 2.1.11 and 2.1.10). This is reminiscent of the recollements obtained in [CJS23] for localizations with respect to arcs.

On the geometric side, we show in section 2.4.2 that the isomorphism classes of graded pinched gentle algebras are in bijection with marked surfaces with conical singularities and admissible graded dissections (see Definition 2.4.8 and Proposition 2.4.10). This is analogous to the correspondence established for gentle algebras in [OPS25, BCS21, PPP19].

Finally, results on derived equivalences of graded gentle algebras given in [HKK17, LP20] allow us to give a class of derived equivalences between graded pinched gentle algebras (Proposition 2.4.15). A more in depth analysis of the role of the graded marked surfaces with conical singularities as a model for the bounded derived category of graded pinched algebras will be made in a future work.

The chapter is structured as follows. In the rest of this section we introduce some definitions and state the main theorems, first for gentle algebras (subsection 2.1.1), then for pinched gentle algebras (subsection 2.1.2). In section 2.2 we recall some constructions in DG categories, such as Drinfeld's DG quotient [Dri04] which will be used in the proof of the main theorem, and we introduce some notations. Some basic definitions and results on spectral sequences are recalled in section 2.3, and applied to describe the first page of a spectral sequence on morphism spaces in DG quotients. In section 2.4 we first show how one can use the surface model of a gentle algebra in order to choose a set of generators of the derived category which will simplify calculations involving a spherical band object. Then we introduce the notion of graded marked surfaces with conical singularities and establish the correspondence with graded pinched gentle algebras. The section 2.5 is devoted to the proof of a technical lemma, namely the formality of the quotient algebra. Finally the proof of the main theorem is made in section 2.6.

2.1.1 Definitions and main results

We first introduce some definitions and state our main result in the more restricted context of graded gentle algebras.

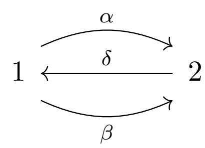
Conventions: A quiver Q is a quadruple $Q = (Q_0, Q_1, \sigma, \tau)$ with finite sets of vertices Q_0 and arrows Q_1, and with $\sigma, \tau : Q_1 \to Q_0$ the source and target functions. We consider right modules with the convention of composition (a path $\beta\alpha$ in a path algebra has source $\sigma(\alpha)$ and target $\tau(\beta)$), and we use the cohomological convention for complexes. For each $a \in Q_0$, let e_a be the path of length zero at a and $P_a = e_a\Lambda$ be the associated indecomposable projective Λ-module. In the rest of the paper K will denote a field of characteristic zero. This hypothesis is used for example in the proof of Proposition 2.5.10, to define a contracting homotopy Ψ.

Definition 2.1.1 *Let $\Lambda = (KQ/\langle I\rangle, |\ .\ |)$ be a $\mathbb{Z}$-graded algebra.*

- *Λ is said to be gentle if:*
  - *Every vertex of Q has at most two incoming and two outgoing arrows,*
  - *I is a set of paths of length two satisfying: for all $\alpha \in Q_1$, there is at most one arrow β such that $0 \neq \alpha\beta \in I$; at most one arrow γ such that $0 \neq \gamma\alpha \in I$; at most one arrow β' such that $0 \neq \alpha\beta' \notin I$; at most one arrow γ' such that $0 \neq \gamma'\alpha \notin I$.*

Suppose now that Λ is gentle.

- *A graded Kronecker of Λ is a pair of arrows (α, β) such that α and β have the same source and the same target, the same degree, and such that they are not loops. It is said to be acyclic if α and β do not belong to an oriented cycle of (Q, I).*
- *let $\omega = \alpha + \mu\beta$ for some graded Kronecker (α, β) of Λ and some $\mu \in K^*$. The localization of Λ at ω is the graded algebra $\Lambda[\omega^{-1}] = (K\widetilde{Q}/\langle\widetilde{I}\rangle, \widetilde{|\ .\ |})$ defined by:*
  - *adding to Q an arrow δ from 2 to 1 of degree $-|\alpha|$:*



  - *adding the relations $\delta\omega - e_1, \omega\delta - e_2$.*

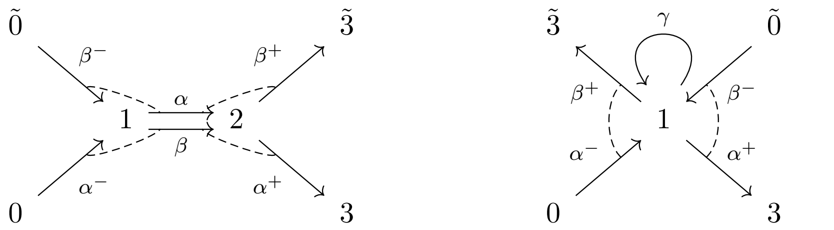
- *Let (α, β) be a graded Kronecker of Λ. Let $1 := \sigma(\alpha) = \sigma(\beta)$, $2 := \tau(\alpha) = \tau(\beta)$, and $\alpha^+, \alpha^-, \beta^+, \beta^-$ be either zero or if they exist, be the (possibly equal) arrows of Q satisfying $\alpha^+\beta, \beta\alpha^-, \beta^+\alpha, \alpha\beta^- \in I$ (see Example 2.1.2 left).*

  The pinching of Λ at (α, β) is the graded algebra $\Lambda_{(\alpha,\beta)} := (KQ'/\langle I' \rangle, |\,.\,|')$ defined by:

  - *removing α and β;*
  - *merging 1 and 2 into a new vertex called 1;*
  - *adding a loop γ of degree 0 at vertex 1;*
  - *setting $|\alpha^+|' = |\alpha^+| + |\alpha|$ and $|\beta^+|' = |\beta^+| + |\beta|$, and leaving unchanged the degree of all the other arrows;*
  - *keeping all relations in I that don't involve α and β, and adding the relations*
    $$\{\alpha^+\beta^-, \beta^+\alpha^-, \beta^+(\gamma + e_1), (\gamma + e_1)\beta^-, \alpha^+(\gamma - e_1), (\gamma - e_1)\alpha^-\}.$$

- *A band object supported by (α, β) is an element of the perfect derived category $per(\Lambda)$ isomorphic to a shift of a twisted complex (see Notations 2.2.1) of the form $(P_2[|\alpha|] \bigoplus P_1[1], \partial = \begin{pmatrix} 0 & \alpha + \mu\beta \\ 0 & 0 \end{pmatrix})$, for some $\mu \in K^*$. We call μ the parameter of the band object.*

Example 2.1.2 *The quiver and relations of a gentle algebra Λ_1 with a graded Kronecker (α, β), and of the associated pinching $\Lambda_{1(\alpha,\beta)}$:*



Λ_1 with graded Kronecker (α,β)

$\Lambda_{1(\alpha,\beta)}$ $\langle \alpha^+\beta^-, \beta^+\alpha^-, \beta^+(\gamma{+}e_1), (\gamma{+}e_1)\beta^-, \alpha^+(\gamma{-}e_1), (\gamma{-}e_1)\alpha^- \rangle$

Recall that graded gentle algebras are in one-to-one correspondence with graded marked surfaces with admissible dissection, and that under this correspondence, graded curves are associated to objects of the bounded derived category (see subsection 2.4.1 and [BCS21, OPS25, PPP19] for more details). Our first main result is the following.

Theorem 2.1.3 *Let Λ' be a graded gentle algebra associated to the graded marked surface with admissible dissection (S, M, Δ, G). Let γ be a simple closed curve on S with winding number zero that does not enclose a subsurface containing only punctures, and let B_γ be an associated band object at the base of a tube.*

(1) There exists a graded gentle algebra Λ and an equivalence $\Psi : per(\Lambda') \to per(\Lambda)$ such that $\Psi(B_\gamma)$ is a band object supported by an acyclic graded Kronecker (α, β) of Λ, with parameter μ.

(2) There is a recollement:

$$\mathcal{D}(\Lambda_{(\alpha,\beta)}) \rightleftarrows \mathcal{D}(\Lambda) \rightleftarrows \mathcal{D}(K[x]/(x^2)) \ ,$$

where x is of degree 1.

(3) There is an equivalence:

$$per(\Lambda')/thick(B_\gamma) \simeq per(\Lambda_{(\alpha,\beta)}).$$

where the left hand side is the Verdier quotient.

Remark 2.1.4 *Theorem 2.1.3 (1) is a reformulation of Corollary 2.4.6, and (2) (resp. (3)) is a direct consequence of Theorem 2.1.11 (resp. Theorem 2.1.10).*

Example 2.1.5 *The algebras Λ_1 and $\Lambda_{1(\alpha,\beta)}$ of Example 2.1.2 give an example of algebras occurring in the recollement of Theorem 2.1.3 (2). An illustration of Theorem 2.1.3 (1) is given by examples 2.4.4 and 2.4.7 by letting $\Lambda' = \Lambda_0$ and $\Lambda = \Lambda_1$.*

2.1.2 Graded pinched gentle algebras

The pinching of a graded gentle algebra at a graded Kronecker leads us to the introduction of the class of graded pinched gentle algebras.

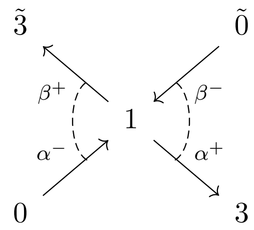
Definition 2.1.6 *A graded K-algebra $\Lambda = (KQ/\langle I\rangle, |\,.\,|)$ is said to be a graded pinched gentle algebra if there is a decomposition $Q_1 = Q_1^g \bigsqcup Q_1^p$ and $I = I^g \bigsqcup I^p$ such that*

- *$(Q^g := (Q_0, Q_1^g, \sigma, \tau), \langle I^g\rangle)$ is a gentle bound quiver. The corresponding gentle algebra $KQ^g/\langle I^g\rangle$ is denoted Λ^g,*
- *Q_1^p is a set of degree zero loops supported on different vertices.*
- *For $v \in Q_0$, the arrows $\alpha_v^-, \alpha_v^+, \beta_v^-, \beta_v^+ \in Q_1^g$ satisfying $\tau(\alpha_v^-) = \sigma(\alpha_v^+) = v = \sigma(\beta_v^+) = \tau(\beta_v^-)$ and $\beta_v^+\alpha_v^-, \alpha_v^+\beta_v^- \in I^g$ (with the possibility of being zero, and with possible compatible identifications between $\{\alpha_v^-, \beta_v^-\}$ and $\{\alpha_v^+, \beta_v^+\}$) can be chosen such that:*

$$I^p = \{\beta_v^+(\gamma_v + e_v), (\gamma_v + e_v)\beta_v^-, \alpha_v^+(\gamma_v - e_v), (\gamma_v - e_v)\alpha_v^- \mid v \in Q_0,\ \gamma_v \in Q_1^p \text{ such that } \sigma(\gamma_v) = v\}.$$

Remark 2.1.7 *The "pinched" or "vanishing" relations of I^p are such that $\forall i \neq j \in Q_0$, $e_j\Lambda e_i \simeq e_j\Lambda^g e_i$. In the special case where α_v^- and α_v^+ (or equivalently β_v^- and β_v^+) are both zero, the pinched relations at v can be seen as gentle by letting $\gamma_v' = \gamma_v + e_v$.*

Example 2.1.8 *The algebra $\Lambda_{1(\alpha,\beta)}$ of Example 2.1.2 is a pinched gentle algebra whose associated gentle algebra $(\Lambda_{1(\alpha,\beta)})^g$ is:*



$(\Lambda_{1(\alpha,\beta)})^g$

The notion of graded Kronecker and pinching extend naturally to the context of graded pinched gentle algebras.

Definition 2.1.9 *Let $\Lambda = (KQ/\langle I\rangle, |\,.\,|)$ be a graded pinched gentle algebra.*

- *A graded Kronecker (α,β) of Λ is a graded Kronecker of the associated gentle algebra Λ^g such that there is no loop $\gamma \in Q_1^p$ based at the source or target of α. It is said to be acyclic if α and β do not belong to an oriented cycle of (Q^g, I^g).*
- *For (α,β) a graded Kronecker of Λ, the pinching of Λ at (α,β) is the graded pinched gentle algebra $\Lambda_{(\alpha,\beta)}$ obtained by performing the same local transformation of (Q,I) as in Definition 2.1.1.*
- *The notion of a band object supported by a graded Kronecker is as in Definition 2.1.1.*

We now state the main results of this paper. Since every graded gentle algebra can be seen as a graded pinched gentle algebra, Theorem 2.1.3 (2) is a particular instance of Theorem 2.1.11, and Theorem 2.1.3 (3) a particular instance of Theorem 2.1.10. Stating the results in this generality also allows us to localize at a collection of disjoint simple closed curves by iterating the process.

Theorem 2.1.10 *Let Λ be a graded pinched gentle algebra and B a band object of $per(\Lambda)$ supported by an acyclic graded Kronecker (α,β), with parameter μ. There is an equivalence:*

$$per(\Lambda)/thick(B) \simeq per(\Lambda_{(\alpha,\beta)}),$$

where the left hand side is the Verdier quotient.

Theorem 2.1.11 *Let Λ be a graded pinched gentle algebra with an acyclic graded Kronecker (α,β). There exists a recollement:*

$$\mathcal{D}(\Lambda_{(\alpha,\beta)}) \leftrightarrows \mathcal{D}(\Lambda) \leftrightarrows \mathcal{D}(K[x]/(x^2)) \ ,$$

where x is of degree 1.

The proofs of Theorem 2.1.10 and Theorem 2.1.11 are made in section 2.6.

2.2 Recollections on DG categories

We introduce notations and recall some properties of DG categories. Our mains references are [CC21], [BK91] and [Dri04].

Notations 2.2.1 *Let $\mathcal{A}$ be a DG category over a base field K.*

• *We denote by $H^*(\mathcal{A})$ its graded homotopy category and by $H^0(\mathcal{A})$ its zeroth homotopy category.*

• *We recall that the category $\mathcal{A}^{pre-tr}$ of one-sided twisted complexes over $\mathcal{A}$ is defined in [BK91] as follow:*

- *Objects in $\mathcal{A}^{pre-tr}$ are formal expressions $(\bigoplus_{i=1}^{n} C_i[r_i], \partial)$ where $n \geq 0$, $C_i \in \mathcal{A}$, $r_i \in \mathbb{Z}$, $\partial = (\partial_{ij})$, $\partial_{ij} \in \mathcal{A}(C_j, C_i)[r_i - r_j]$ is homogeneous of degree 1, $\partial_{ij} = 0$ for $i \geq j$ and $d_{naive}\partial + \partial^2 = 0$, where $d_{naive}\partial := (d_{\mathcal{A}}(\partial_{ij}))$.*

- *$f = (f_{ij}) \in \mathcal{A}^{pre-tr}((\bigoplus_{j=1}^{n} C_j[r_j], \partial), (\bigoplus_{i=1}^{m} C'_i[r'_i], \partial'))$ verify $f_{ij} \in \mathcal{A}(C_j, C'_i)[r'_i - r_j]$ and the composition is the matrix multiplication.*

- *The differential is defined by $df := d_{naive}f + \partial' f - (-1)^l f\partial$ if deg$f_{ij} = l$.*

The DG category $\mathcal{A}$ can be seen as a full DG subcategory of $\mathcal{A}^{pre-tr}$. For $f : X \to Y$ a closed morphism of degree 0 of $\mathcal{A}$, let $Cone(f)$ be the object $(Y \bigoplus X[1], \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}) \in \mathcal{A}^{pre-tr}$.

Let $C = (\bigoplus_{j=1}^{n} C_j[r_j], \partial), C' = (\bigoplus_{i=1}^{m} C'_i[r'_i], \partial') \in \mathcal{A}^{pre-tr}$. A homogeneous basis $\{b_h^{kl}\}_{1 \leq h \leq n^{kl}}$ of $\mathcal{A}(C_l, C'_k)$ for each $l \in \{1, ..., n\}$ and $k \in \{1, ..., m\}$ induces a basis $\{[b_h^{kl}] \mid 1 \leq k \leq m$, and $1 \leq l \leq n$, and $1 \leq h \leq n^{kl}\}$ of $\mathcal{A}^{pre-tr}(C, C')$, where $[b_h^{kl}]_{ij} = b_h^{kl}$ if $i = k$ and $j = l$, and zero otherwise. The degrees in $\mathcal{A}^{pre-tr}$ are then $|[b_h^{kl}]|^{pre-tr} = |b_h^{kl}| + r_l - r'_k$.

We will extend this bracket notation to linear combinations and sometimes drop them when there is no ambiguity and the context is clear.

• *The triangulated category $\mathcal{A}^{tr}$ associated to $\mathcal{A}$ is by definition $H^0(\mathcal{A}^{pre-tr})$. The distinguished triangles are isomorphic to diagrams $X \to Y \to Cone(f) \to X[1]$ with $f : X \to Y$ a degree 0 closed morphism of $\mathcal{A}$. Any inclusions $\mathcal{A} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{A}^{pre-tr}$ induce a triangulated equivalence $\mathcal{A}^{tr} \simeq \mathcal{C}^{tr}$.*

Example 2.2.2 *Let $\Lambda := (KQ/\langle I\rangle, |\,.\,|, d)$ be a DG algebra given by a DG quiver with relations. We view Λ as a DG category $\mathcal{P}(\Lambda)$ with objects $\mathcal{P}(\Lambda)_0 = Q_0$ and morphism spaces $\mathcal{P}(\Lambda)(i, j) = e_j\Lambda e_i$, with the induced grading and differential. Using [CC21](Rem 6.2.5 (2)), one can see that the split closure of the triangulated category $\mathcal{P}(\Lambda)^{tr}$ is equivalent to the perfect derived category $per(\Lambda)$.*

Drinfeld introduced and studied in [Dri04] a notion of DG quotient.

Definition 2.2.3 *[Dri04](§3.1) Let $\mathcal{A}$ be a DG category and $\mathcal{B}$ a full subcategory. The quotient DG category $\mathcal{A}/\mathcal{B}$ has the same objects as $\mathcal{A}$. It has a new morphism $\epsilon_U : U \to U$ of degree -1 for all object U of $\mathcal{B}$, and all of its compositions with existing morphisms. Formally, one has an isomorphism of vector spaces (but not of complexes),*

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y) \leftarrow \bigoplus_{n\in\mathbb{N}} \mathrm{Hom}^{\langle n\rangle}_{\mathcal{A}/\mathcal{B}}(X,Y)$$

where $\mathrm{Hom}^{\langle n\rangle}_{\mathcal{A}/\mathcal{B}}(X,Y)$ is the direct sum, over all families $(U_i)_{1\leq i\leq n}$ of objects in $\mathcal{B}$, of tensor products $\mathrm{Hom}_{\mathcal{A}}(U_n,U_{n+1})\otimes K[1]\otimes \mathrm{Hom}_{\mathcal{A}}(U_{n-1},U_n)\otimes ... \otimes K[1]\otimes \mathrm{Hom}_{\mathcal{A}}(U_0,U_1)$ where $U_{n+1}=Y, U_0=X$ and $K[1]$ is the complex with only K in degree -1.

If ϵ is the canonical generator of $K[1]$, the application sends the product $f_n\otimes\epsilon\otimes ...\otimes\epsilon\otimes f_0$ to the composition $f_n\epsilon_{U_n}f_{n-1}...\epsilon_{U_1}f_0$. The differential is given by the Leibniz rule and $d(\epsilon_U)=id_U$ for all U in $\mathcal{B}$. In particular $\mathrm{Hom}^{\langle 0\rangle}_{\mathcal{A}/\mathcal{B}}(X,Y)=\mathrm{Hom}_{\mathcal{A}}(X,Y)$.

This DG quotient is an enhancement of the triangulated quotient:

Theorem 2.2.4 *[Dri04](Th 3.4) There is a triangulated equivalence:*

$$(\mathcal{A}/\mathcal{B})^{tr} \simeq \mathcal{A}^{tr}/\mathcal{B}^{tr}.$$

2.3 Spectral sequences and Drinfeld quotients

In this section, $\mathcal{B}$ will denote a full subcategory of a DG category $\mathcal{A}$. We will present how the Hom spaces of $\mathcal{A}/\mathcal{B}$ naturally inherit a filtration which gives rise to a spectral sequence that can be used to compute the Hom spaces of $H^*\mathcal{A}/\mathcal{B}$. Our main reference in this section is McCleary's book *A user's guide to spectral sequences* [McC00].

Using the notation of Definition 2.2.3, for all objects X,Y in $\mathcal{A}$, the natural filtration

$$0=F^{-1}C\subset ...\subset F^{p-1}C\subset F^pC\subset ...\subset C:=\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)$$

given by $F^pC\simeq \bigoplus_{n=0}^{p}\mathrm{Hom}^{\langle n\rangle}_{\mathcal{A}/\mathcal{B}}(X,Y)$ is:

- *Compatible with the differential*: the differential d of C restricts to a map $d: F^pC\to F^pC$ for each $p\in\mathbb{Z}$,
- *Exhaustive*: $C=\bigcup_s F^sC$,
- *Bounded below*: for each n, there is a value $s(n)$ with $F^{s(n)}C^n=\{0\}$.

These data allow us to construct a spectral sequence:

Theorem 2.3.1 *[McC00](Th - 2.6) The filtration F of C determines a spectral sequence $\{E_r^{*,*},d_r\}$ starting on page 1 with d_r of bidegree $(-r,r+1)$ and*

$$E_1^{p,q}\simeq H^{p+q}(F^pC/F^{p-1}C)$$

This spectral sequence is build recursively starting at $r = 0$. Let $E_0^{p,q} = F^pC^{p+q}/F^{p-1}C^{p+q}$ and $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ be the differential induced by the quotient. Suppose that $E_r^{p,q}$ and the differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p-r,q+r+1}$ are given. By definition of a spectral sequence, there is an isomorphism $E_{r+1}^{p,q} \simeq \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d_r^{p+r,q-r-1})$. One can then verify that the differential d induces a differential $d_{r+1}^{p,q} : E_{r+1}^{p,q} \to E_{r+1}^{p-(r+1),q+(r+1)+1}$.

The filtration F also induces a filtration of the homology $H(C,d)$ in the following way:

$$F^pH(C,d) = \mathrm{Im}\big(H(\iota) : H(F^pC,d) \to H(C,d)\big)$$

where ι is the inclusion. Recall the following definition:

Definition 2.3.2 *[McCoo](Def 2.4) A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to converge to a graded R-module H^* if there is a filtration F on H^* such that*

$$E_\infty^{p,q} \simeq E^{p,q}(H^*, F),$$

where $E_\infty^{p,q}$ is the limit term of the spectral sequence, and $E^{p,q}(H^*,F)$ the bigraded module given by

$$E^{p,q}(H^*,F) = F^pH^{p+q}/F^{p-1}H^{p+q}.$$

The following theorem applies:

Theorem 2.3.3 *[McCoo](Th 3.2) The spectral sequence of Theorem 2.3.1 converges to $H(C,d)$, that is,*

$$E_\infty^{p,q} \simeq F^pH^{p+q}(C,d)/F^{p-1}H^{p+q}(C,d).$$

Since we are working over a field, all short exact sequences split and the convergence implies that for all p and q,

$$H^{p+q}(\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)) \simeq \bigoplus_{k\in\mathbb{Z}} E_\infty^{p+k,q-k}. \tag{2.1}$$

Note that $\forall p < 0, \forall q \in \mathbb{Z}$, $E_0^{p,q} \simeq 0$. The following proposition will allow us to compute easily the first terms $E_1^{p,q}$.

Proposition 2.3.4 *Let $\mathcal{B}$ be a full subcategory of a DG category $\mathcal{A}$ over a field K, and let X, Y be two objects of $\mathcal{A}$. Let $\{E_r^{*,*}, d_r\}$ be the spectral sequence associated to the natural filtration of $C = \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)$. Then*

$$E_1^{p,q} \simeq \bigoplus_{(U_i)\in\mathcal{B}} \bigoplus_{\substack{k_1+\ldots+k_{p+1}\\ = 2p+q}} H^{k_{p+1}}(\mathrm{Hom}_{\mathcal{A}}(U_p,Y)) \otimes K[1] \otimes H^{k_p}(\mathrm{Hom}_{\mathcal{A}}(U_{p-1},U_p)) \otimes \ldots$$
$$\ldots \otimes K[1] \otimes H^{k_1}(\mathrm{Hom}_{\mathcal{A}}(X,U_1)).$$

Proof: By definition, $E_1^{p,q} = H^{p+q}(F^pC/F^{p-1}C)$. Moreover $F^pC/F^{p-1}C$ is isomorphic to

$$\bigoplus_{(U_i)\in\mathcal{B}} \mathrm{Hom}_{\mathcal{A}}(U_p,Y) \otimes K[1] \otimes \mathrm{Hom}_{\mathcal{A}}(U_{p-1},U_p) \otimes \ldots \otimes K[1] \otimes \mathrm{Hom}_{\mathcal{A}}(X,U_1)$$

as a graded vector space. We will show that $F^pC/F^{p-1}C$ is actually isomorphic to it as a complex, endowed with the differential coming from the tensor product. Recall that for complexes (C^r, d^r), ..., (C^1, d^1), their tensor product $C^r \otimes ... \otimes C^1$ has for differential $d^{\otimes}$, sending an homogeneous element $x_r \otimes ... \otimes x_1$ to

$$d^{\otimes}(x_r \otimes ... \otimes x_1) = \sum_{k=1}^{r} (-1)^{\sum\limits_{l=k+1}^{r} deg(x_l)} x_r \otimes ... \otimes d_k(x_k) \otimes ... \otimes x_1.$$

On one hand for $f_p \otimes \epsilon_p \otimes f_{p-1} \otimes ... \otimes \epsilon_1 \otimes f_0$ a homogeneous element in

$$(\bigoplus_{(U_i)\in\mathcal{B}} \mathrm{Hom}_{\mathcal{A}}(U_p, Y) \otimes K[1] \otimes \mathrm{Hom}_{\mathcal{A}}(U_{p-1}, U_p) \otimes ... \otimes K[1] \otimes \mathrm{Hom}_{\mathcal{A}}(X, U_1), d^{\otimes}),$$

one has

$$\begin{aligned} &d^{\otimes}(f_p \otimes \epsilon_p \otimes f_{p-1} \otimes ... \otimes \epsilon_1 \otimes f_0) \\ &\quad= \sum_{k=0}^{p} (-1)^{\sum\limits_{l=k+1}^{p} deg(f_l)-(p-k)} f_p \otimes \epsilon_p \otimes ... \otimes d(f_k) \otimes ... \otimes \epsilon_1 \otimes f_0 \\ &\quad+ \sum_{k=1}^{p} (-1)^{\sum\limits_{l=k}^{p} deg(f_l)-(p-k)} f_p \otimes \epsilon_p \otimes ... \otimes d(\epsilon_k) \otimes ... \otimes \epsilon_1 \otimes f_0 \\ &\quad= \sum_{k=0}^{p} (-1)^{\sum\limits_{l=k+1}^{p} deg(f_l)-(p-k)} f_p \otimes \epsilon_p \otimes ... \otimes d(f_k) \otimes ... \otimes \epsilon_1 \otimes f_0 \end{aligned}$$

since $K[1]$ is endowed with the zero differential. On the other hand, for a homogeneous element $\overline{f_p\epsilon_p f_{p-1}...\epsilon_1 f_0}$ of $F^pC/F^{p-1}C$,

$$\begin{aligned} d(\overline{f_p\epsilon_p f_{p-1}...\epsilon_1 f_0}) &= \sum_{k=0}^{p} (-1)^{\sum\limits_{l=k+1}^{p} deg(f_l)-(p-k)} \overline{f_p\epsilon_p...d(f_k)...\epsilon_1 f_0} \\ &+ \sum_{k=1}^{p} (-1)^{\sum\limits_{l=k}^{p} deg(f_l)-(p-k)} \overline{f_p\epsilon_p...f_k f_{k-1}...\epsilon_1 f_0} \\ &= \sum_{k=0}^{p} (-1)^{\sum\limits_{l=k+1}^{p} deg(f_l)-(p-k)} \overline{f_p\epsilon_p...d(f_k)...\epsilon_1 f_0} \end{aligned}$$

since the $f_p\epsilon_p...f_k f_{k-1}...\epsilon_1 f_0$ are in $F^{p-1}C$. This shows that the differential induced by the quotient coincides with the one coming from the tensor product, inducing an isomorphism of complex.

For the moment we have shown an isomorphism of complexes

$$\begin{aligned} E_1^{p,q} \simeq H^{p+q}(&\bigoplus_{(U_i)\in\mathcal{B}} \mathrm{Hom}_{\mathcal{A}}(U_p, Y) \otimes K[1] \otimes \mathrm{Hom}_{\mathcal{A}}(U_{p-1}, U_p) \otimes ... \\ &... \otimes K[1] \otimes \mathrm{Hom}_{\mathcal{A}}(X, U_1), d^{\otimes}). \end{aligned}$$

We conclude using the Künneth formula for complexes [Wei94](Th 3.6.3). Given two complexes P and Q, there exist for all n a short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(P) \otimes H^j(Q) \to H^n(P \otimes Q) \to \bigoplus_{i+j=n-1} Tor_1^K(H^i(P), H^j(Q)) \to 0.$$

Since we are working with vector spaces, the torsion groups vanish and there is an isomorphism $\bigoplus_{i+j=n} H^i(P) \otimes H^j(Q) \simeq H^n(P \otimes Q)$. This gives

$$E_1^{p,q} \simeq \bigoplus_{(U_i)\in\mathcal{B}} \bigoplus_{\substack{h_1+...+h_{2p+1} \\ =p+q}} H^{h_{2p+1}}(\mathrm{Hom}_{\mathcal{A}}(U_p, Y)) \otimes H^{h_{2p}}(K[1]) \otimes H^{h_{2p-1}}(\mathrm{Hom}_{\mathcal{A}}(U_{p-1}, U_p)) \otimes ...$$
$$... \otimes H^{h_2}(K[1]) \otimes H^{h_1}(\mathrm{Hom}_{\mathcal{A}}(X, U_1)).$$

Finally since $H^i(K[1])$ is non zero if and only if $i = -1$, this gives the desired formula.

□

2.4 Admissible dissections and graded pinched gentle algebras

2.4.1 Adapted admissible dissections

In order to simplify computations in $per(\Lambda)$, it will be useful to choose an adapted set of generators. Recall that graded gentle algebras are in one-to-one correspondence with marked surfaces with graded admissible dissection, that is, a quadruple (S, M, Δ, G) where S is a compact oriented surface with boundary, M a finite set of ○ and ● marked points and punctures, Δ an admissible ○-dissection, and G a grading of the minimal oriented intersections. See for example [HKK17, OPS25, APS23, LP20] for the statement of this result, and for more details and examples. Throughout this section, we will use the terminology of [CJS23](section 1.5). Contrary to the convention adopted in [APS23], we will draw marked surfaces with admissible dissection in such a way that the orientation of the arrows of the corresponding quiver will be given by rotating clockwise around a ○-point.

By a theorem of [LP20] this correspondence gives rise, in the homologically smooth case (that is, in the case without ●-punctures), to an equivalence between the derived category of the graded gentle algebra and the partially wrapped Fukaya category $\mathcal{W}(S, M, \eta(\Delta, G))$ of the associated marked surface with graded admissible dissection:

Theorem 2.4.1 *[LP20](Theorem 3.11) For a homologically smooth graded gentle algebra Λ with associated marked surface with graded admissible dissection (S, M, Δ, G), there is a line field $\eta(\Delta, G)$ on S such that there is an equivalence*

$$per(\Lambda) \simeq \mathcal{W}(S, M, \eta(\Delta, G))$$

Remark 2.4.2 *In case Λ is not homologically smooth, following [OZ22](Definition 3.24, Remark 3.26) we can still embed $per(\Lambda)$ into a partially wrapped Fukaya category $\mathcal{W}(S_s, M_s, \eta(\Delta_s, G_s))$ as follows. If Λ is* *associated to (S, M, Δ, G), let S_s be the marked surface obtained by replacing each* $\color{red}\bullet$*-puncture by a boundary component containing one* $\color{red}\bullet$*-point and one* $\color{green}\circ$*-point, and let M_s be the new set of marked points and punctures. The collection Δ induces an arc system on S_s that can be completed into a full arc system Δ_s by choosing a new* $\color{green}\circ$*-arc for each new boundary component. Finally a choice of grading G_s that specializes to G on Δ induces an equivalence between $per(\Lambda)$ and the full subcategory of $\mathcal{W}(S_s, M_s, \eta(\Delta_s, G_s))$ generated by Δ. Equivalences for $\mathcal{W}(S_s, M_s, \eta(\Delta_s, G_s))$ induce equivalences for $per(\Lambda)$ (see [OZ22], Lemma 3.28).*

For $\gamma : \mathbb{S}^1 \to S$ an immersed curve, we denote by $w_{\eta(\Delta,G)}(\gamma)$ its winding number with respect to $\eta(\Delta, G)$ (see [LP20] Definition 1.1.3).

Following [OPS25](Assumption 2.7), we assume that any finite collection of curves is in minimal position, that is, the number of intersections of each pair of (not necessarily distinct) curves in this set is minimal in their respective homotopy class. According to [Thu08], it follows from [FHS82] and [Neu01] that, up to homotopy, this assumption is always satisfied. Given two curves γ, γ', denote by $|\gamma \cap \gamma'|$ this minimal number of intersections.

For $\gamma : \mathbb{S}^1 \to S$ a simple closed curve, let $D_\gamma : S \to S$ be the associated Dehn twist along γ (see for instance [FM12] for a definition). For δ a $\color{green}\circ$-arc on (S, M) intersecting a simple closed curve γ, each (possibly equal) end point of δ give rise to an oriented intersection between δ and $D_\gamma(\delta)$, denoted $\alpha(\delta)$ and $\beta(\delta)$. Recall that an oriented intersection between to consecutive $\color{green}\circ$-arcs at a $\color{green}\circ$-point is called a *minimal oriented intersection*.

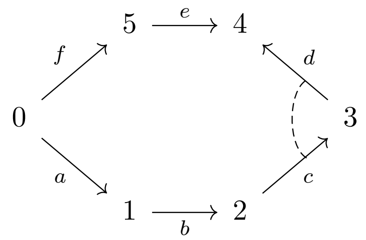
Proposition 2.4.3 *Let (S, M, Δ, G) be a graded marked surface and let $\gamma_1, ..., \gamma_r$ be a collection of pairwise non-intersecting simple closed curves on S with zero winding number. Suppose moreover that these curves do not enclose a subsurface containing only punctures.*

There exists a graded admissible $\color{green}\circ$*-dissection (Δ', G') on (S, M) such that $\eta(\Delta', G') \simeq \eta(\Delta, G)$, and for all $i \in \{1, ..., r\}$ there exists $\delta_i \in \Delta'$ such that:*

- *The endpoints of δ_i are not* $\color{green}\circ$*-punctures,*
- $|\delta_i \cap \gamma_i| = 1$*,*
- $D_{\gamma_i}(\delta_i) \in \Delta'$*,*
- *For all $\delta \in \Delta' \backslash \{\delta_i, D_{\gamma_i}(\delta_i)\}$, $|\delta \cap \gamma_i| = 0$,*
- *$\alpha(\delta_i)$ and $\beta(\delta_i)$ are minimal and $G'(\alpha(\delta_i)) = G'(\beta(\delta_i))$.*

We will call such a dissection (Δ', G') an *admissible dissection adapted to* the collection $\{\gamma_1, ..., \gamma_r\}$.

Example 2.4.4 *Consider the graded gentle algebra Λ_0 given by:*



and $|a| = |d| = |f| = 0$, $|c| = 1$ *and* $|b| = |e| = -1$. *The associated graded marked surface* S *is depicted on the left of the following figure:*

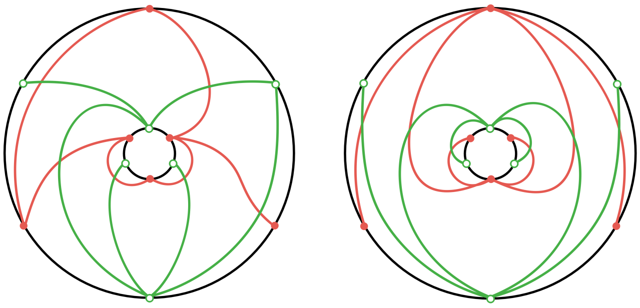


Figure 2.4.1: A marked surface with a non-adapted admissible dissection (left) and with an adapted admissible dissection (right).

The figure on the right is an example of adapted admissible dissection (the grading being zero) to the unique simple closed curve of S. *Let* S_1 *be this new graded marked surface.*

The proof of Proposition 2.4.3 will rely on the following lemma:

Lemma 2.4.5 *[LP20](Remark 1.2.5) Let* S *be an oriented surface with non-empty boundary* ∂S *decomposing as the disjoint union of connected components* $\partial S = \bigsqcup_{i=1}^{r} \partial_i S$.

A line field η *on* S *relates its genus* g *with the winding number of its boundary components via the formula:*

$$\sum_{i=1}^{r} w_{\eta(\Delta,G)}(\partial_i S) = 4 - 2r - 4g$$

Proof of Proposition 2.4.3: Each simple closed curve γ_i has zero winding number, regardless of the choice of orientation.

Let S' be the surface obtained by cutting S along each γ_i. It is the disjoint union of connected surfaces $S' = \bigsqcup_{j=1}^{s} S'_j$, and each γ_i give rise to two boundary components of S': γ_i^+ and γ_i^-. Let us show by contradiction that each S'_j contains a boundary component of the original surface S (and thus contains at least one $\circ$ marked point on this boundary).

Since by hypothesis each S'_j cannot contain only punctures, each S'_j containing a puncture must contain a boundary component of S.

Suppose now that S'_j does not contain any puncture and that all its distinct boundary components $\rho_1^j, \ldots, \rho_{r_j}^j$ are in $\{\gamma_1^+, \ldots, \gamma_r^+, \gamma_1^-, \ldots, \gamma_r^-\}$. Let g_j be its genus. By Lemma 2.4.5, after choosing the orientation on each ρ_i^j which is compatible with the orientation of S'_j, we have

$$\sum_{i=1}^{r_j} w_{\eta(\Delta,G)}(\rho_i^j) = 4 - 2r_j - 4g_j \Leftrightarrow 0 = 4 - 2r_j - 4g_j$$

$$\Leftrightarrow r_j = 2 \text{ and } g_j = 0,$$

but then ρ_1^j and ρ_2^j should be homotopic, a contradiction. Thus we can choose an arbitrary $\circ$ marked point m_j in each S'_j.

For all $j \in \{1, ..., s\}$, and for all $k \in \{1, ..., r_j\}$, let δ_k^j be a simple arc from $\delta_k^j(0) = m_j$ to $\delta_k^j(1) = \rho_k^j(1)$ (after the identification with some $\gamma_i^\pm : \mathbb{S}^1 \to S$), and such that $\forall k \neq k' \in \{1, ..., r_j\}$, $|\delta_k^j \cap \delta_{k'}^j| = 0$.

For each $i \in \{1, ..., r\}$, this gives rise to a $\circ$-arc δ_i on S intersecting γ_i once (and no other $\gamma_{i'\neq i}$) once and transversely, in the following way: if $\gamma_i^+ = \rho_k^j$ and $\gamma_i^- = \rho_{k'}^{j'}$, define δ_i to be the concatenation $\delta_i = \delta_k^j.(\delta_{k'}^{j'})^{-1}$. Moreover, define $\mu_i^+ = \delta_k^j.\gamma_i.(\delta_k^j)^{-1}$ and $\mu_i^- = \delta_{k'}^{j'}.\gamma_i.(\delta_{k'}^{j'})^{-1}$. Cutting S along the $\mu_i^\pm$, we obtain a surface $\Gamma = \Gamma' \bigsqcup (\bigsqcup_{i=1}^{r} \Gamma_i)$, where each Γ_i is an annulus containing γ_i and whose boundary components $\mu_i^\pm$ each have one $\circ$ marked point. The following figure depicts the situation:

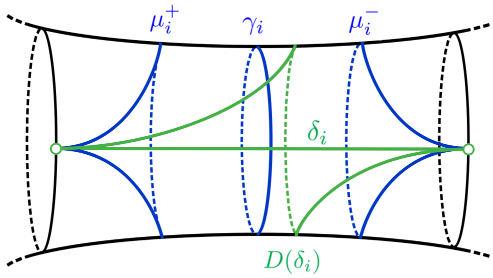


Figure 2.4.2: Arc configuration in the neighbourhood of a simple closed curve.

Let $\Gamma' = \bigsqcup_{k \in K_0} \Gamma_k$ be the decomposition of Γ' into connected components. We have seen above that for each k in K_0, Γ_k contains at least one $\circ$ marked point, thus [HKK17](Lemma 3.3) ensures that it admits an admissible dissection A_k. We set $\Delta' := (\bigsqcup_{k \in K_0} A_k) \bigsqcup \{\delta_1, ..., \delta_r, D_{\gamma_1}(\delta_1), ..., D_{\gamma_r}(\delta_r)\}$. In order to see that this indeed give an admissible dissection on (S, M), we need to check that each pair of arcs are non-intersecting, which is the case by construction, that they are pairwise distinct and that they cut S into polygons, each of which containing exactly one $\bullet$-point. We call such a polygon an admissible polygon.

The curves in $\Delta' \bigsqcup \{\gamma_1, ..., \gamma_r\}$ are in minimal position by assumption. For $k \neq k' \in K_0$, any $\delta \in A_k$ and $\delta' \in A_{k'}$ are distinct since they do not share common endpoints. For all i, δ_i is distinct from $D_{\gamma_i}(\delta_i)$ and for $i \neq j \in \{1, ..., r\}$, $\delta_i \not\simeq \delta_j, \delta_i \not\simeq D_{\gamma_j}(\delta_j)$ and $D_{\gamma_i}(\delta_i) \not\simeq D_{\gamma_j}(\delta_j)$ since they do not have the same minimal relative position with respect to the set $\{\gamma_1, ..., \gamma_r\}$.

Let S_0 be a subsurface cut out by Δ'. If all boundary arcs of S_0 are contained in some A_k, then S_0 is an admissible polygon since A_k is an admissible collection. Otherwise S_0 is obtained by gluing an admissible polygon of some A_k with a collection of triangles of the form $(\mu_i^\pm, \delta_i, D_{\gamma_i}(\delta_i))$. The gluing is made along the $\mu_i^\mp$'s, and since elements of $\{\delta_1, ..., \delta_r, D_{\gamma_1}(\delta_1), ..., D_{\gamma_r}(\delta_r)\}$ are all distinct, S_0 is a polygon. Finally since each triangle $(\mu_i^\pm, \delta_i, D_{\gamma_i}(\delta_i))$ contains no $\bullet$ points, S_0 is admissible.

Now $\eta(\Delta, G)$ induces a unique grading G' on Δ' such that $\eta(\Delta', G') \simeq \eta(\Delta, G)$, and by construction $\alpha(\delta_i)$ and $\beta(\delta_i)$ are minimal for all i. The fact that $G'(\alpha(\delta_i)) = G'(\beta(\delta_i))$ follows from $w_{\eta(\Delta,G)}(\gamma_i) = 0$. $\square$

Proposition 2.4.3 has the following consequence on the algebraic side:

Corollary 2.4.6 *Let $\Lambda = (KQ/I, |\,.\,|)$ be a graded gentle algebra associated to the graded marked surface with admissible dissection (S, M, Δ, G).*

Let $B_1, ..., B_r$ be a collection of band objects of $per(\Lambda)$ of parameters $(n_i = 1, \mu_i \in K^*)$ corresponding to a collection $\gamma_1, ..., \gamma_r$ of simple closed curves on S under the equivalence $per(\Lambda) \simeq \mathcal{W}(S, M, \eta(\Delta, G))$ or under the inclusion into $\mathcal{W}(S_s, M_s, \eta(\Delta_s, G_s))$ in case Λ is not homologically smooth. Suppose that the γ_i's are pairwise non-intersecting and do not enclose a subsurface containing only punctures.

There exists a graded gentle algebra $\Lambda' = (KQ'/I', |\,.\,|')$ and an equivalence $\Psi : per(\Lambda) \to per(\Lambda')$ such that each $\Psi(B_i)$ is a band object supported by an acyclic graded Kronecker (α_i, β_i) of Λ', and such that $\forall i \neq j$, $\{\sigma(\alpha_i), \tau(\alpha_i)\} \bigcap \{\sigma(\alpha_j), \tau(\alpha_j)\} = \emptyset$.

Example 2.4.7 *Let Λ_0 be the gentle algebra of Example 2.4.4, and B be a band object of parameter $(1, \mu)$ of $per(\Lambda)$ corresponding to the only simple closed curve of S. Then Theorem 2.4.1 shows that Λ_0 is derived equivalent to the gentle algebra associated the graded marked surface S_1. This algebra is the gentle algebra Λ_1 of Example 2.1.2.*

Under this equivalence, B is isomorphic to (a shift of) $(P_2 \bigoplus P_1[1], \partial = \begin{pmatrix} 0 & \alpha + \mu\beta \\ 0 & 0 \end{pmatrix})$.

Proof of Corollary 2.4.6: A choice of admissible dissection (Δ', G') as in 2.4.3 gives the desired algebra Λ', which is derived equivalent to Λ by [HKK17](Proposition 3.2). The acyclic graded Kroneckers come from the minimal angles $\{\alpha(\delta_i), \beta(\delta_i)\}$, and the description of the B_i's in $per(\Lambda')$ in term of one sided twisted complexes is given by [HKK17](Theorem 4.3).

□

2.4.2 Graded marked surfaces with conical singularities

In this section we establish the correspondence between graded pinched gentle algebras and marked surfaces with conical singularities and simple admissible dissections, a generalisation of graded marked surface to surfaces with conical singularities.

Definition 2.4.8

- *A (topological) oriented surface with conical singularities is a compact topological space S in which every point admits a neighbourhood homeomorphic to one of the following:*

  - *The open unit disk D of $\mathbb{R}^2$,*
  - *The half disk $D \cap H$, where H is the closed upper half plane of $\mathbb{R}^2$,*
  - *The wedge sum $C_0 = D \sqcup D' / \sim$ of two pointed open disks (see the left-hand side of Figure 2.4.3).*

  Since S is compact, it admits a finite number of boundary components, each homeomorphic to a circle, and a finite set of points C, called the set of conical singularities, that correspond to the singular point of C_0. The surface $S \backslash C$ must be oriented. Elements of C will be represented by symbols ⊙.

- *A marked surface with conical singularities $\mathcal{S} = (S, C, M \sqcup P)$ is the data of:*
  - *A surface with conical singularities (S, C),*
  - *A finite set $P = P_\circ \sqcup P_\bullet$ of points in the interior of $S \backslash C$, called punctures and represented respectively by symbols $\circ$ and $\bullet$,*
  - *A finite set $M = M_\circ \sqcup M_\bullet$ of points in the boundary of $S \backslash C$, called marked points and represented respectively by symbols $\circ$ and $\bullet$. They are required to alternate on each boundary component, and each component must contain at least one marked point.*
- *A $\circ$-arc is a non contractible curve on S with endpoints in $M_\circ \sqcup P_\circ$. Every curve will be considered up to homotopy. Note that $\circ$-arcs are allowed to go through conical singularities.*
- *Let $\mathcal{S}_{/\!/}$ be the marked surface obtained from $\mathcal{S}$ by splitting each conical singularity into two $\circ$-punctures (see the right-hand side of Figure 2.4.3). A simple admissible dissection Δ on $\mathcal{S}$ is a set of $\circ$-arcs that don't intersect (in the interior of S) and that satisfy:*
  - *For each $c \in C$ there is exactly one arc γ_c in Δ that goes through c, it goes through it only once and does not go through other singularities,*
  - *The collection $\Delta_{/\!/}$ of arcs on $\mathcal{S}_{/\!/}$ induced from Δ by splitting each γ_c in two, is an admissible dissection.*
- *A grading G on a simple admissible dissection Δ is the data of an integer for each minimal oriented angle of Δ.*

Remark 2.4.9 *There are two other equivalent ways of thinking of marked surfaces with conical singularities.*

1. *First we can think of a marked surface with conical singularities as a marked surface where pairs of $\circ$-punctures have been identified. More precisely, given a marked surface $(S, M \sqcup P)$ and a fixed-point free involution ι on a subset $P_\circ{}^\iota$ of $P_\circ$, the associated marked surface with conical singularities $\mathcal{S}$ is obtained by taking the quotient of S under the identifications $p \sim \iota(p)$ for all $p \in P_\circ{}^\iota$. Thus the set of conical singularities is the quotient of $P_\circ{}^\iota$ under the action of ι, and the new set of $\circ$-punctures is $P_\circ \backslash P_\circ{}^\iota$. This construction is inverse to the splitting of the conical singularities $\mathcal{S}_{/\!/}$, where ι is given by remembering each pairing of new $\circ$-punctures.*

   Under this correspondence, simple admissible dissections of the marked surface with conical singularities $\mathcal{S}$ are in bijection with admissible dissections on $(S, M \sqcup P)$ satisfying that each puncture in $P_\circ{}^\iota$ is of valency one. Under this correspondence, the $\circ$-arcs γ_p and $\gamma_{\iota(p)}$ admitting respectively p and $\iota(p)$ in $P_\circ{}^\iota$ as one of their end points, give rise via concatenation to an arc going through the corresponding conical singularity.

   Note that here we must exclude the case where $(S, M \sqcup P)$ is a sphere without boundary, with one $\bullet$-puncture and two $\circ$-punctures p and p', and with $\iota(p) = p'$, since the $\circ$-arc in the admissible dissection of $(S, M \sqcup P)$ would descend to a closed curve on $\mathcal{S}$. This is the only case where this can happen.

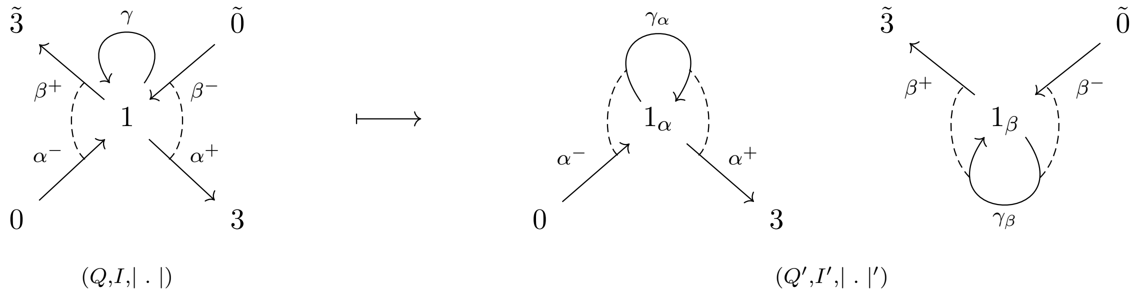
   Each grading G on $(\mathcal{S}, \Delta)$ corresponds to a grading $G_{/\!/}$ on $(\mathcal{S}_{/\!/}, \Delta_{/\!/})$ that associate zero to each minimal oriented angle at $\circ$-punctures arising from conical singularities.

2. *Similarly each collection of non intersecting simple closed curves on a marked surface $(S, M \sqcup P)$ gives rise to a marked surface with conical singularities by taking the topological quotient that identifies each simple closed curve with a point.*

The next proposition follows easily from the correspondence established in [HKK17, BCS21, PPP19, LP20] for gentle algebras. A quiver with relation (Q, I) as in Definition 2.1.6 will be called a pinched gentle quiver.

Proposition 2.4.10 *Graded pinched gentle quivers are in one to one correspondence with marked surfaces with conical singularities endowed with a graded simple admissible dissection.*

Proof: Let $(Q, I, |\ .\ |)$ be a graded pinched gentle quiver with set of vanishing loops Q_1^p. Let γ_v be a loop in Q_1^p attached at vertex v, and let $\alpha^+, \alpha^-, \beta^+, \beta^-$ be the (possibly identified or equal to zero) arrows given in Definition 2.1.6. By Remark 2.1.7, we can assume that at least one of α^+, α^-, and at least one of β^+, β^- is non-zero. Let $(Q', I', |\ .\ |')$ be the graded quiver with relations obtained by resolving each loop in Q_1^p in the following way:



Each loop γ in Q_1^p gives rise to two degree zero loops γ_α and γ_β attached on different vertices, and satisfying the relations $\alpha^+\gamma_\alpha = \gamma_\alpha\alpha^- = 0 = \beta^+\gamma_\beta = \gamma_\beta\beta^-$. Thus $(Q', I', |\ .\ |')$ is a graded gentle quiver and thus corresponds to a marked surface with graded admissible dissection $(S, M \sqcup P, \Delta, G)$. Moreover for each γ in Q_1^p, the set of couples $(\gamma_\alpha, \gamma_\beta)$ give the data of a partial coupling on the $\circ$-punctures of $(S, M \sqcup P)$. By Remark 2.4.9 (1), this gives rise to a unique marked surfaces with conical singularities with graded simple admissible dissection.

Conversely, starting from a marked surfaces $\mathcal{S}$ with conical singularities and graded simple admissible dissection, one can split its conical singularities to obtain $\mathcal{S}_{/\!/}$ as in Remark 2.4.9 (1). One gets an associated graded gentle quiver and can do the inverse procedure of the last figure to obtain a graded pinched gentle quiver. By definition the two constructions are inverse from each other.

□

Example 2.4.11 *Consider the pinched gentle algebra $\Lambda_{1(\alpha,\beta)}$ of Example 2.1.2 (it corresponds to the left quiver with relations of the last figure). The corresponding marked surface with conical singularities, depicted on the left part of the following figure, is homeomorphic to the conical singularity C_0. The right part illustrates the marked surface with admissible dissection obtained by splitting the conical singularity.*

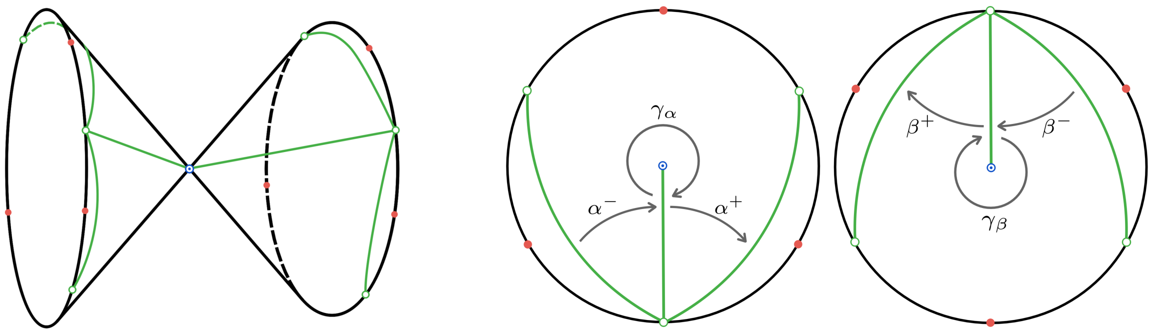


Figure 2.4.3: A marked surface with conical singularities and graded simple admissible dissection, and the splitting of its conical singularity.

Remark 2.4.12 *Simple admissible dissections are a special case of a more general notion of admissible dissection on marked surfaces with conical singularities, introduced in Definition 3.3.1. It includes dissections admitting several arcs going through a conical singularity.*

The localization described in Theorem 2.1.11 corresponds at the level of surfaces with conical singularities to the contraction of the simple closed curve. More precisely, we have the following proposition:

Proposition 2.4.13 *Let Λ be a graded pinched gentle algebra with a graded Kronecker (α, β), let $\mathcal{S}$ be its associated marked surface with conical singularities and let γ be the simple closed curve on $\mathcal{S}$ corresponding to (α, β). (Note that by definition of a graded Kronecker, γ does not go through a conical singularity).*

The marked surface with conical singularities $\mathcal{S}_{(\alpha,\beta)}$, associated to the pinching $\Lambda_{(\alpha,\beta)}$ of Λ at (α, β), is homeomorphic to the quotient of $\mathcal{S}$ that identifies all points in γ.

Proof: The marked surface with conical singularities obtained by identifying all points in γ is the same as the one obtained by:

(i) Splitting $\mathcal{S}$ at γ to obtain a marked surface with conical singularities with new boundary components γ_α and γ_β,

(ii) contracting γ_α and γ_β to obtain two $\circ$-punctures p_α and p_β,

(iii) gluing p_α and p_β to create a conical singularity.

Splitting the other conical singularities of the marked surface with conical singularities $\mathcal{S}'$ obtained at step (ii) yields exactly the one used in the proof of Proposition 2.4.10 to define $\mathcal{S}_{(\alpha,\beta)}$.

□

Remark 2.4.14 *Up to derived equivalence we can suppose that (α, β) is in degree zero. In this case, the pinching of $\mathcal{S}$ at γ sends the graded simple admissible dissection associated to Λ to the graded simple admissible dissection associated to $\Lambda_{(\alpha,\beta)}$ (with the identification of the arcs corresponding to $\sigma(\alpha)$ and $\tau(\alpha)$).*

We now state a useful result on derived equivalences for graded pinched gentle algebras. For a closed curve γ on a marked surface with conical singularities with simple admissible dissection $(\mathcal{S}, \Delta, G)$ that does not go through a singularity, we define its winding number to be $w_{\eta(\Delta_{/\!/}, G_{/\!/})}(\gamma)$.

Proposition 2.4.15 *Let Λ be a graded pinched gentle algebra corresponding to a marked surface with conical singularities with graded simple admissible dissection $(\mathcal{S}, \Delta, G)$. Let γ be a simple closed curve on $\mathcal{S}$ with zero winding number that does not go through a singularity nor encloses a subsurface containing only punctures or singularities.*

There exists a graded simple admissible dissection (Δ', G') on $\mathcal{S}$ that is adapted to γ (see Proposition 2.4.3), and such that $per(\Lambda) \simeq per(\Lambda')$ where Λ' is the graded pinched gentle algebra associated to $(\mathcal{S}, \Delta', G')$.

We can then use this new description of the perfect derived category to apply Theorem 2.1.10 and take the pinching of Λ' at the graded Kronecker (α, β) corresponding to γ. By Proposition 2.4.13, $\Lambda'_{(\alpha,\beta)}$ gives a graded simple admissible dissection on the marked surface with conical singularities obtained by contracting γ in $\mathcal{S}$.

Proof: Let Γ be the graded gentle algebra obtained by replacing each vanishing loop δ_i in Q_1^p by a zero graded Kronecker (α_i, β_i) in such a way that Γ satisfies $\Gamma_{(\alpha_i,\beta_i)_i} = \Lambda$. By Theorem 2.1.10, each choice of band objects B_i supported on (α_i, β_i) gives an equivalence $per(\Lambda) \simeq per(\Gamma)/thick(B_i)_i$.

Let $(\mathcal{S}(\Gamma), \Delta(\Gamma), G(\Gamma))$ be the marked surface with admissible dissection associated to Γ, and for all i let γ_i be the simple closed curve corresponding to (α_i, β_i). The curve γ is lifted to a curve on $\mathcal{S}$ with zero winding number, and it does not intersect the γ_i's. Moreover this collection of closed curves does not enclose a subsurface containing only punctures, so we can apply Proposition 2.4.3. Let $(\Delta(\Gamma)', G(\Gamma)')$ be the resulting graded admissible dissection, let Γ' be the associated graded gentle algebra, and let $(\alpha'_i, \beta'_i)_i, (\alpha, \beta)$ be the new graded Kroneckers.

Setting $\Lambda' = \Gamma'_{(\alpha'_i,\beta'_i)_i}$ we get $per(\Lambda') \simeq per(\Gamma')/thick(B_i)_i \simeq per(\Gamma)/thick(B_i)_i \simeq per(\Lambda)$. By Remark 2.4.14, the graded simple admissible dissection associated to Λ' corresponds to the image of $(\Delta(\Gamma)', G(\Gamma)')$ under the pinching at each γ_i, and thus is adapted to γ.

□

2.5 Formality of the quotient

In this section we place ourselves in the following setting:

Setting 2.5.1 *Let $\Lambda = (KQ/\langle I\rangle, |\,.\,|)$ be a graded pinched gentle algebra (see Definition 2.1.6). Let B be a band object of $\mathcal{P}(\Lambda)^{pre-tr}$ (see Example 2.2.2), of parameter $\mu \in K^*$ and supported by an acyclic graded Kronecker (α, β) of Λ (see Definition 2.1.9). We set $\omega := \alpha + \mu\beta$ and $1 := \sigma(\alpha), 2 := \tau(\alpha)$.*

Let $\mathcal{A}$ be the full subcategory of $\mathcal{P}(\Lambda)^{pre-tr}$ supported on $\mathcal{P}(\Lambda)_0$ and B. The full subcategory $\mathcal{B}$ of $\mathcal{A}$ supported on B satisfies the following property:

Theorem 2.5.2 *[BK91](§4 - Th 1) Let $\langle B\rangle$ be the smallest strictly full triangulated subcategory of the triangulated category $\mathcal{A}^{tr} \simeq per(\Lambda)$, which contains B. There is an equivlance:*

$$\mathcal{B}^{tr} \simeq \langle B\rangle.$$

We will show that $(\mathcal{A}/\mathcal{B})_\circ$, the full subcategory of $\mathcal{A}/\mathcal{B}$ supported on $\mathcal{P}(\Lambda)_0$, is formal.

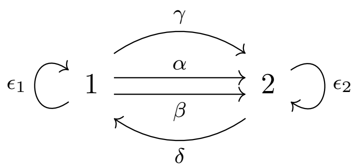
2.5.1 A quasi-equivalence

Lemma 2.5.3 *There is a DG equivalence*

$$(\mathcal{A}/\mathcal{B})_\circ \simeq \mathcal{P}(K\widehat{Q}/\langle I\rangle, \widehat{|\,.\,|}, \widehat{d})$$

where:

- $(K\widehat{Q}/\langle I\rangle, \widehat{|\,.\,|})$ *is obtained by adding to* Q *an arrow* γ *from 1 to 2 of degree* $|\alpha| - 2$*, loops* ϵ_1 *and* ϵ_2 *at 1 and 2 of degree* -1*, and an arrow* δ *from 2 to 1 of degree* $-|\alpha|$*:*



- $\widehat{d}$ *is defined by setting* $\widehat{d}(\epsilon_1) = \delta\omega - e_1$, $\widehat{d}(\epsilon_2) = (-1)^q(e_2 - \omega\delta)$, $\widehat{d}(\gamma) = (-1)^{q+1}(\omega\epsilon_1 + \epsilon_2\omega)$*, and by sending every other arrows to zero.*

Proof: Recall that ϵ is the new generator added to $\mathrm{End}_{\mathcal{A}}(B)$ when taking the quotient. Let $i, j \in Q_0$. Using the bracket of Notation 2.2.1, let

$$[e_1) := [e_1] \in \mathrm{Hom}_{\mathcal{A}}(P_1, B), \ [e_2) := [e_2] \in \mathrm{Hom}_{\mathcal{A}}(P_2, B),$$
$$(e_1] := [e_1] \in \mathrm{Hom}_{\mathcal{A}}(B, P_1) \text{ and } (e_2] := [e_2] \in \mathrm{Hom}_{\mathcal{A}}(B, P_2).$$

Every morphism $\rho = \begin{pmatrix}\rho_2\\ \rho_1\end{pmatrix} \in \mathrm{Hom}_{\mathcal{A}}(P_i, B)$ factors as $\rho = (e_2]\rho_2 + (e_1]\rho_1$ and dually for a morphism $\rho' \in \mathrm{Hom}_{\mathcal{A}}(B, P_j)$. The equivalence is then given by sending $\mathrm{Hom}_\Lambda(P_i, P_j)$ to itself via $\Lambda \hookrightarrow K\widehat{Q}/\langle I\rangle$ for all i, j, and $(e_1]\epsilon[e_1)$ to ϵ_1, $(e_2]\epsilon[e_2)$ to ϵ_2, $(e_1]\epsilon[e_2)$ to δ and $(e_2]\epsilon[e_1)$ to γ.

□

Definition 2.5.4 *Recall that the localization* $\Lambda[\omega^{-1}]$ *of* Λ *at* ω *has been introduced in Definition 2.1.1. Using notations of Lemma 2.5.3, we define the morphism of DG algebras* $\phi : (K\widehat{Q}/\langle I\rangle, \widehat{|\,.\,|}, \widehat{d}) \to \Lambda[\omega^{-1}]$ *in the following way:*

Let $\phi : Q_0 \sqcup \widehat{Q}_1 \to K\widetilde{Q}$ *be given by* $\phi_{|Q_0 \sqcup Q_1 \sqcup \{\delta\}} = id$, $\phi(\epsilon_1) = \phi(\epsilon_2) = \phi(\gamma) = 0$*. It extends to an epimorphism of graded algebra* $\phi : K\widehat{Q} \to K\widetilde{Q}$*, and to an epimorhism* $\phi : (K\widehat{Q}/\langle I\rangle, \widehat{|\,.\,|}, \widehat{d}) \to \Lambda[\omega^{-1}]$ *of DG algebra since* $\phi(I) \subset \widetilde{I}$ *and* $\phi(\widehat{d}(\epsilon_1)), \phi(\widehat{d}(\epsilon_2)), \phi(\widehat{d}(\gamma)) \in \langle \widetilde{I}\rangle$.

The next subsection will be devoted to the proof of the following proposition.

Proposition 2.5.5 *The functor* $\mathcal{P}(\phi) : (\mathcal{A}/\mathcal{B})_\circ \to \mathcal{P}(\Lambda[\omega^{-1}])$ *induced by the morphism* ϕ *of Definition 2.5.4 is a quasi-equivalence.*

2.5.2 Computation of $\mathrm{H}^*(\mathcal{A}/\mathcal{B})_\circ$

Lemma 2.5.6 $H^*\mathrm{End}_{\mathcal{A}}(B) \simeq K[x]/(x^2)$*, where x is of degree 1*

Proof: For a graded pinched gentle algebra $\Lambda = (KQ/\langle I\rangle, |\ .\ |)$ and $i \neq j \in Q_0$, a basis of $e_j\Lambda e_i \simeq e_j\Lambda^g e_i$ is given by paths in (Q^g, I^g). Let us give a basis of $e_1\Lambda e_2$ in case it is non-empty.

First there can be no path from 2 to 1 of the form $\rho'\alpha\rho$ or $\rho'\beta\rho$. Indeed, if $\rho'\alpha\rho \in e_1\Lambda e_2$ then the acyclicity of (α,β) ensures that $\rho\alpha \in I^g$ and $\alpha\rho' \in I^g$, but then $\rho\beta \notin I^g$ and $\beta\rho' \notin I^g$, and thus $\rho'\alpha\rho\beta$ is an oriented cycle, a contradiction.

Now let ρ be a non-zero path in $e_1\Lambda e_2$ not passing through α or β. We can suppose without loss of generality that $\rho\alpha \notin I^g$. The acyclicity of (α,β) implies that $\alpha\rho \in I^g$ and thus $\beta\rho \notin I^g$.

There can be no other path in $e_1\Lambda e_2$ different from ρ. Indeed suppose that ρ' is such a path. Then $\rho'\beta \in I^g$, otherwise $\rho'\beta\rho \in e_1\Lambda e_2$ is a contradiction. Since in a gentle bound quiver, every path γ belongs to a unique maximal path $\hat{\gamma}$, we have $\hat{\rho} = \hat{\alpha} = \hat{\rho'}$. But then we can suppose that $\rho = \rho''\rho'$, and thus $\rho = \rho'''\alpha\rho'$ or $\rho = \rho'''\beta\rho'$, a contradiction.

In conclusion, if $e_1\Lambda e_2 \neq \emptyset$ then $e_1\Lambda e_2 = \langle\rho\rangle$ where ρ is a path which we can suppose verify $\rho\alpha, \beta\rho \notin I$. For the rest this proof, let $\rho = 0$ if $e_1\Lambda e_2$ is zero.

We can deduce:

$$e_1\Lambda e_1 = \langle e_1, \rho\alpha\rangle,\ \ e_2\Lambda e_1 = \langle \alpha, \beta, \beta\rho\alpha\rangle,\ \ e_1\Lambda e_2 = \langle\rho\rangle,\ \ e_2\Lambda e_2 = \langle e_2, \beta\rho\rangle, \tag{2.2}$$

which gives the basis $\mathrm{End}_{\mathcal{A}}(B) = \langle [e_1], [e_2], [\alpha], [\beta], [\rho], [\rho\alpha], [\beta\rho], [\beta\rho\alpha]\rangle$, with $|[e_1]| = |[e_2]| = 0, |[\alpha]| = |[\beta]| = 1, |[\rho]| = |\rho| + q - 1$.

Dropping without ambiguity the bracket notation, the differential is given by:

$$\begin{aligned} d(e_1) = \omega = -d(e_2), \qquad & d(\rho) = \omega\rho - (-1)^{|[\rho]|}\rho\omega = \mu\beta\rho - (-1)^{|[\rho]|}\rho\alpha, \\ d(\alpha) = d(\beta) = 0, \qquad & d(\beta\rho) = (-1)^{|[\rho]|}\beta\rho\alpha,\ \ d(\rho\alpha) = \mu\beta\rho\alpha,\ \ d(\beta\rho\alpha) = 0, \end{aligned}$$

which gives the decomposition as complexes

$$\mathrm{End}_{\mathcal{A}}(B) \ \simeq\ \langle [e_1], [e_2], [\alpha], [\beta]\rangle \bigoplus \langle [\rho], [\rho\alpha], [\beta\rho], [\beta\rho\alpha]\rangle \ =:\ E_0 \bigoplus E_1.$$

Now E_1 is acyclic since $\mathrm{Ker}\, d_{|E_1} = \langle \rho\alpha - (-1)^{|[\rho]|}\mu\beta\rho\rangle \bigoplus \langle\beta\rho\alpha\rangle = \mathrm{Im}\, d_{|E_1}$, and

$$H^*\mathrm{End}_{\mathcal{A}}(B) \ \simeq\ H^*E_0 \ \simeq\ \langle id = e_1 + e_2\rangle^0 \bigoplus \langle \alpha, \beta \mid \omega\rangle^1. \tag{2.3}$$

□

A direct computation shows that:

Lemma 2.5.7 *Using the notations of (2.2),*

- $H^*\mathrm{Hom}_{\mathcal{A}}(P_1, B) \simeq \langle \alpha, \beta | \omega\rangle$,
- $H^*\mathrm{Hom}_{\mathcal{A}}(P_2, B) \simeq \langle e_2\rangle$,

- $H^*\mathrm{Hom}_{\mathcal{A}}(B, P_1) \simeq \langle e_1 \rangle$,
- $H^*\mathrm{Hom}_{\mathcal{A}}(B, P_2) \simeq \langle \alpha, \beta | \omega \rangle$.

Lemma 2.5.8 *For all $i \in Q_0 \backslash \{1,2\}$, $H^*\mathrm{Hom}_{\mathcal{A}}(P_i, B) \simeq 0 \simeq H^*\mathrm{Hom}_{\mathcal{A}}(B, P_i)$.*

Proof: Let $e_1 \Lambda e_i \simeq e_1 \Lambda^g e_i = \langle \rho_1, ..., \rho_r \rangle$ and $e_2 \Lambda e_i \simeq e_2 \Lambda^g e_i = \langle \gamma_1, ..., \gamma_s \rangle$ be two bases where each ρ_k, γ_l is a path in (Q^g, I^g). We use the same notation to designate the induced basis of $\mathrm{Hom}_{\mathcal{A}}(P_i, B)$.

Recall that B is of the form $(P_2[|\alpha|] \bigoplus P_1[1], \partial = \begin{pmatrix} 0 & \alpha + \mu\beta \\ 0 & 0 \end{pmatrix})$ for some $\mu \in K^*$, and thus for f in $\mathrm{Hom}_{\mathcal{A}}(P_i, B)$ the differential is given by $df = (\alpha + \mu\beta) f$ (see Notations 2.2.1).

Since $i \neq 1$, one can choose an ordering of the ρ_l's and $r' \in \{0, ..., r\}$ such that $\forall l \in \{1, ..., r'\}$, $\beta\rho_l \in I$ and $\forall l \in \{r'+1, ..., r\},\ \alpha\rho_l \in I$. So for $(\lambda_l), (\mu_h) \in K^{r+s}$,

$$\begin{aligned} d(\sum_{l=1}^{r} \lambda_l \rho_l + \sum_{h=1}^{s} \mu_h \gamma_h) = 0 \ &\Leftrightarrow\ (\alpha + \mu\beta)(\sum_{l=1}^{r'} \lambda_l \rho_l + \sum_{r'+1}^{r} \lambda_l \rho_l) + (\alpha + \mu\beta)(\sum_{h=1}^{s} \mu_h \gamma_h) = 0 \\ &\Leftrightarrow\ \sum_{l=1}^{r'} \lambda_l \alpha \rho_l + \sum_{r'+1}^{r} \lambda_l \mu \beta \rho_l = 0 \\ &\Leftrightarrow\ \forall l \in \{1, ..., r\},\ \lambda_l = 0, \end{aligned}$$

where the last equivalence comes from the fact that in a gentle algebra distinct paths correspond to distinct elements, and thus the $\alpha\rho_l$'s and the $\beta\rho_l$'s are pairwise distinct. We deduce that the kernel is $\mathrm{Ker}\ d = \langle \gamma_1, ..., \gamma_s \rangle$.

Now since $i \neq 2$, one can choose an ordering of the γ_h's and $s' \in \{0, ..., s\}$ such that

$$\begin{aligned} &\forall h \in \{1, ..., s'\}, \exists l_h \in \{1, ..., r\},\ \gamma_h = \alpha\rho_{l_h}, \text{ and} \\ &\forall h \in \{s'+1, ..., s\}, \exists l_h \in \{1, ..., r\},\ \gamma_h = \beta\rho_{l_h}. \end{aligned}$$

Thus for all $(\mu_h) \in K^s$,

$$\sum_{h=1}^{s} \mu_h \gamma_h = \sum_{h=1}^{s'} \mu_h \alpha \rho_{l_h} + \sum_{h=s'+1}^{s} \frac{\mu_h}{\mu} \mu \beta \rho_{l_h} = d(\sum_{h=1}^{s'} \mu_h \rho_{l_h} + \sum_{h=s'+1}^{s} \frac{\mu_h}{\mu} \rho_{l_h}).$$

We can deduce $\mathrm{Ker}\ d \simeq \mathrm{Im}\ d$ and $H^*\mathrm{Hom}_{\mathcal{A}}(P_i, B) = 0$. The other case is dual.

□

Corollary 2.5.9 *If i or j does not belong to $\{1,2\}$, then $H^*\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_i, P_j) \simeq \mathrm{Hom}_{\Lambda}(P_i, P_j)$.*

Proof: Let (i,j) be such a pair. By Equation 2.1, $H^k\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_i, P_j) \simeq \bigoplus_{l \in \mathbb{Z}} E_\infty^{k-l,l}$. Let $q \in \mathbb{Z}$. Since $H^*\mathrm{Hom}_{\mathcal{A}}(P_i, B) \simeq 0$ or $H^*\mathrm{Hom}_{\mathcal{A}}(B, P_j) \simeq 0$ by Lemma 2.5.8, Proposition 2.3.4 shows that $\forall p > 0$, $E_1^{p,q} \simeq 0$. Thus $\forall p \neq 0$, $E_\infty^{p,q} \simeq 0$. Moreover since for all $r > 1$ the codomain of $d_r^{0,q}$ and the domain of $d_r^{r,q-r-1}$ are zero, we have $E_\infty^{0,q} \simeq E_1^{0,q} \simeq \mathrm{Hom}_{\Lambda}(P_i, P_j)^q$.

□

Proposition 2.5.10 *Let* $(i,j) \in \{(1,1),(1,2),(2,1),(2,2)\}$ *and let* $E^{*,*}$ *be the spectral sequence on the space* $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_i,P_j)$*. Let*

$$H^*\mathrm{Hom}_{\mathcal{A}}(P_i,B) =: \langle t\rangle, H^*\mathrm{Hom}_{\mathcal{A}}(B,P_j) =: \langle z\rangle \text{ and } H^*\mathrm{End}_{\mathcal{A}}(B) =: \langle id_B\rangle \bigoplus \langle y\rangle$$

be the basis given in Lemma 2.5.7 and in (2.3), and let $a := |z|+|t|-1$.

- $\forall q \in \mathbb{Z}$, $E_\infty^{0,q} \simeq \mathrm{Hom}_\Lambda(P_i,P_j)^q$,
- $\forall p > 0$, $E_\infty^{p,a-p} \simeq \langle z\epsilon(y\epsilon)^{p-1}t\rangle$,
- $\forall p \neq 0, \forall q \neq a-p$, $E_\infty^{p,q} \simeq 0$.

Proof: We have the following relations: $zy = yt = y^2 = zt = 0$. Let $p > 0$. For $q \in \mathbb{Z}$, since by Proposition 2.3.4

$$E_1^{p,q} \simeq \bigoplus_{\substack{k_1+\ldots+k_{p+1}\\ =2p+q}} H^{k_{p+1}}(\mathrm{Hom}_{\mathcal{A}}(B,P_j)) \otimes K[1] \otimes H^{k_p}(\mathrm{End}_{\mathcal{A}}(B)) \otimes \ldots \otimes K[1] \otimes H^{k_1}(\mathrm{Hom}_{\mathcal{A}}(P_i,B)),$$

we have

$$\begin{aligned} E_1^{p,q} \neq 0 &\Leftrightarrow \exists (k_i) \in \{0,1\}^{p-1},\ |z| + \sum_i k_i + |t| = 2p+q \\ &\Leftrightarrow 0 \le 2p+q-|z|-|t| \le p-1 \\ &\Leftrightarrow a+1-2p \le q \le a-p. \end{aligned}$$

For such p and q, a basis of $E_1^{p,q}$ is given by

$$E_1^{p,q} = \langle \{z\epsilon^{n_0}\Pi_{l=1}^r(y\epsilon^{n_l})t \mid r \ge 0,\ \forall l \in \{0,\ldots,r\}.\ n_l \ge 1,\ \sum_{l=0}^r n_l = p \text{ and } |z|+r-p+|t| = p+q\}\rangle,$$

where ϵ is a generator of $K[1]$. Moreover,

$$\begin{aligned} d_1^{p,q}(z\epsilon^{n_0}\Pi_{l=1}^r(y\epsilon^{n_l})t) &= \sum_{i=0}^r (-1)^{|z(\Pi_{l=1}^i \epsilon^{n_l-1}y)|} z(\Pi_{l=1}^i \epsilon^{n_l-1}y) d(\epsilon^{n_i})(\Pi_{l=i+1}^r y\epsilon^{n_l})t \\ &= \sum_{\substack{i=0\\ n_i\neq 1 \text{ odd}}}^r (-1)^{|z(\Pi_{l=1}^i \epsilon^{n_l-1}y)|} z(\Pi_{l=1}^i \epsilon^{n_l-1}y)\epsilon^{n_i-1}(\Pi_{l=i+1}^r y\epsilon^{n_l})t, \end{aligned}$$

since $\forall n \ge 1.\ d(\epsilon^n) = \sum_{i=1}^n (-1)^{|\epsilon^{i-1}|}\epsilon^{n-1} = 0$ if n even and ϵ^{n-1} otherwise. We first compute the second page of the spectral sequence.

For $q \in \mathbb{Z}$, $d_1^{0,q} = 0$ and $E_1^{1,q} \neq 0 \Leftrightarrow q = a-1$. Since $d_1^{1,a-1}(z\epsilon t) = (-1)^{|z|}zt = 0$, $E_2^{0,q} \simeq E_1^{0,q}$.

For all $p > 0$, $d_1^{p,a-p}(z\epsilon(y\epsilon)^{p-1}t) = 0$, and for all $\rho = z\epsilon^{n_0}\Pi_{l=1}^r(y\epsilon^{n_l})t \in E_1^{p+1,a-p-2}$, one has $d_1^{p+1,a-p-2}(\rho) = 0$ since $n_l = 1$ or 2. Thus $E_2^{p,a-p} \simeq E_1^{p,a-p}$.

We regroup the remaining cases by complexes. For each $b > 0$, let $C_b^\bullet$ be defined by $\forall i \geq a,\ C_b^i = 0$, $\forall i < a,\ C_b^i = E_1^{b+(a-i),a-b-2(a-i)}$ and $\forall i < a,\ d_C^i = d_1^{b+(a-i),a-b-2(a-i)}$. In this way, $\forall p > 0$ and $\forall q \in \{a+1-2p, ..., a-p-1\}$, $E_1^{p,q}$ is equal to C_{2p+q-a}^{p+q}. Let us show that each $C_b^\bullet$ is acyclic by defining morphisms $\Psi : C_b^\bullet \to C_b^\bullet[-1]$ satisfying $d_{C_b^\bullet} \circ \Psi + \Psi \circ d_{C_b^\bullet} = id_{C_b^\bullet}$. For $i < a$ and $\rho = z\epsilon^{n_0}\Pi_{l=1}^r(y\epsilon^{n_l})t$ a basis element of C_b^i, let

$$X = \{l \in \{0, ..., r\} |\ n_l \text{ is odd and different from } 1\} \text{ and } Y = \{l \in \{0, ..., r\} |\ n_l \text{ is even}\}.$$

Let $\lambda = \frac{1}{|X|+|Y|}$ and define Ψ on paths by

$$\Psi(\rho) = \lambda \sum_{i \in Y} (-1)^{|z(\Pi_{l=1}^i \epsilon^{n_l - 1} y)|} z(\Pi_{l=1}^i \epsilon^{n_l - 1} y)\epsilon^{n_i+1}(\Pi_{l=i+1}^r y\epsilon^{n_l})t.$$

Then ρ appears with coefficient $\lambda|Y|$ in $d \circ \Psi(\rho)$ and with coefficient $\lambda|X|$ in $\Psi(\rho) \circ d$. For any other path $\rho' \neq \rho$ appearing with coefficient μ in $d \circ \Psi(\rho)$, it appears with coefficient $-\mu$ in $\Psi \circ d(\rho)$ and vice-versa. Thus $d \circ \Psi + \Psi \circ d = \rho$ holds.

Finally the values of E_∞ correspond to the second page since $\forall p, q \in \mathbb{Z}, \forall r \geq 2, d_r^{p,q} = 0.$

□

Proof of Proposition 2.5.5: Since $\mathcal{P}(\Lambda[\omega^{-1}])$ has zero differential, it suffices to show that the morphism $H^*\mathcal{P}(\phi) : H^*\mathrm{Hom}_{(\mathcal{A}/\mathcal{B})_\circ}(P_i, P_j) \to \mathrm{Hom}_{\mathcal{P}(\Lambda[\omega^{-1}])}(P_i, P_j)$ is an isomorphism for all i, j.

If i or j is different from 1 or 2, the relations in $\Lambda[\omega^{-1}]$ are such that $e_j\Lambda[\omega^{-1}]e_i \simeq e_j\Lambda e_i$. Thus $H^*\mathcal{P}(\phi)$ is an isomorphism by Corollary 2.5.9.

For $(i, j) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, Equation 2.1 and Proposition 2.5.10 give the isomorphism:

$$H^*\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_i, P_j) \simeq \mathrm{Hom}_\Lambda(P_i, P_j) \bigoplus \langle z\epsilon(y\epsilon)^n t \mid n \geq 0\rangle.$$

Let us consider the case $(i, j) = (1, 1)$. On one hand, the relations in $\Lambda[\omega^{-1}]$ allow to choose the following basis: $\mathrm{Hom}_{\mathcal{P}(\Lambda[\omega^{-1}])}(P_1, P_1) = \mathrm{Hom}_\Lambda(P_1, P_1) \bigoplus \langle \delta(\alpha\delta)^n\alpha \mid n \geq 0\rangle$. On the other hand, if we choose the following representatives: $t = \alpha, y = \alpha, z = e_1$, the equivalence of Lemma 2.5.3 gives the rewriting $z\epsilon(y\epsilon)^n t = e_1\epsilon(e_2\alpha e_1\epsilon)^n e_2\alpha = \delta(\alpha\delta)^n\alpha$, and $H^*\mathcal{P}(\phi)$ is an isomorphism.

Similarly for the other cases we found:

$$H^*\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_2, P_2) = \mathrm{Hom}_\Lambda(P_2, P_2) \bigoplus \langle \alpha\delta(\alpha\delta)^n | \ n \geq 0\rangle,$$
$$H^*\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_1, P_2) = \mathrm{Hom}_\Lambda(P_1, P_2) \bigoplus \langle \alpha\delta(\alpha\delta)^n\alpha | \ n \geq 0\rangle,$$
$$H^*\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_2, P_1) = \mathrm{Hom}_\Lambda(P_2, P_1) \bigoplus \langle \delta(\alpha\delta)^n | \ n \geq 0\rangle,$$

which are isomorphic under $H^*\mathcal{P}(\phi)$ to the corresponding spaces in $\mathcal{P}(\Lambda[\omega^{-1}])$.

□

2.6 Proof of the main results

We now prove Theorem 2.1.10 and 2.1.11. For this we will need two lemmas.

Lemma 2.6.1 *Let Λ be the quotient of a path algebra KQ by a finitely generated ideal $I = \langle \rho_1, ..., \rho_r \rangle$. Let $\overline{Q}_0 \subseteq Q_0$ be a subset of vertices and $e = \sum_{i \in \overline{Q}_0} e_i$ the associated idempotent.*

There is an isomorphism $[[-]] : eKQe \to K\overline{Q}$, where $\overline{Q}$ is the quiver whose set of vertices is $\overline{Q}_0$, and whose arrows from i to j are symbols $[\alpha_r...\alpha_1]$, where $\alpha_r...\alpha_1$ is a path from i to j in Q which passes through no other vertices of $\overline{Q}_0$.

It induces an isomorphism $e\Lambda e \simeq K\overline{Q}/\overline{I}$, where $\overline{I} = \langle [[\delta\rho_i\gamma]] \mid \forall i \in \{1, ..., r\} \text{ and } \delta \in T,\ \gamma \in S \rangle$, with:

- $T = \{\alpha_r...\alpha_1 \mid r \geq 1,\ \forall i.\alpha_i \in Q_1 \text{ and } \tau(\alpha_r) \in \overline{Q}_0 \text{ and } \forall i.\sigma(\alpha_i) \in Q_0 \backslash \overline{Q}_0\} \bigsqcup \{e_i \mid i \in \overline{Q}_0\}$, *and dually*
- $S = \{\alpha_r...\alpha_1 \mid r \geq 1,\ \forall i.\alpha_i \in Q_1 \text{ and } \sigma(\alpha_1) \in \overline{Q}_0 \text{ and } \forall i.\tau(\alpha_i) \in Q_0 \backslash \overline{Q}_0\} \bigsqcup \{e_i \mid i \in \overline{Q}_0\}$.

Note that $\overline{Q}$ can have an infinite set of arrows, and that $\overline{I}$ is not necessarily admissible.

Lemma 2.6.2 *Let $\Lambda = (KQ/\langle I \rangle, |\ .\ |)$ be a graded pinched gentle algebra, (α, β) be a graded Kronecker in Q as in Definition 2.1.9, and $\omega = \alpha + \mu\beta$ for some $\mu \in K^*$. There are equivalences:*

$$\mathcal{D}(\Lambda[\omega^{-1}]) \simeq \mathcal{D}(\Lambda_{(\alpha,\beta)}) \textit{ and } per(\Lambda[\omega^{-1}]) \simeq per(\Lambda_{(\alpha,\beta)}).$$

Proof: Since P_1 and P_2 become isomorphic in $\Lambda[\omega^{-1}] =: K\widetilde{Q}/\langle \widetilde{I} \rangle$, $\mathcal{D}(\Lambda[\omega^{-1}]) \simeq \mathcal{D}(e\Lambda[\omega^{-1}]e)$ and $per(\Lambda[\omega^{-1}]) \simeq per(e\Lambda[\omega^{-1}]e)$ where $e = \sum_{l \in Q_0 \backslash \{2\}} e_l$. One can compute $e\Lambda[\omega^{-1}]e =: K\overline{Q}/\overline{I}$ using Lemma 2.6.1. First we have:

$$\overline{Q}_1 = \{[\rho] \mid \rho \in \widetilde{Q}_1 \text{ and } \widetilde{\sigma}(\rho) \neq 2 \neq \widetilde{\tau}(\rho)\} \bigsqcup \{[\beta^+\beta], [\alpha^+\alpha], [\delta\alpha], [\delta\beta]\}$$

and Lemma 2.6.1 gives the following description of $\overline{I}$:

$$\begin{aligned}
\overline{I} =& \langle [[\delta\rho_i\gamma]] \mid \rho_i \in \widetilde{I} = I^g \sqcup I^p \sqcup \{\delta\omega - e_1, \omega\delta - e_2\} \text{ and } \delta \in T,\ \gamma \in S \rangle \\
=& \langle \{[\beta_s...\beta_1][\alpha_r...\alpha_1] \mid [\beta_s...\beta_1], [\alpha_r...\alpha_1] \in \overline{Q}_1 \text{ and } \beta_1\alpha_r \in I^g\} \\
&\sqcup \{[\beta_s...\beta_1]([\gamma_v] + e_v) \mid [\beta_s...\beta_1] \in \overline{Q}_1 \text{ and } \beta_1(\gamma_v + e_v) \in I^p\} \\
&\sqcup \{([\gamma_v] + e_v)[\beta_s...\beta_1] \mid [\beta_s...\beta_1] \in \overline{Q}_1 \text{ and } (\gamma_v + e_v)\beta_s \in I^p\} \\
&\sqcup \{[\alpha_r...\alpha_1]([\gamma_v] - e_v) \mid [\alpha_r...\alpha_1] \in \overline{Q}_1 \text{ and } \alpha_1(\gamma_v - e_v) \in I^p\} \\
&\sqcup \{([\gamma_v] - e_v)[\alpha_r...\alpha_1] \mid [\alpha_r...\alpha_1] \in \overline{Q}_1 \text{ and } (\gamma_v - e_v)\alpha_s \in I^p\} \\
&\sqcup \{[[\delta\omega - e_1]] = [\delta\alpha] + \mu[\delta\beta] - e_1, \\
&\quad [[\alpha^+(\omega\delta - e_2)\alpha]] = [\alpha^+\alpha][\delta\alpha] - [\alpha^+\alpha],\ [[\alpha^+(\omega\delta - e_2)\beta]] = [\alpha^+\alpha][\delta\beta], \\
&\quad [[\beta^+(\omega\delta - e_2)\alpha]] = \mu[\beta^+\beta][\delta\alpha],\ [[\beta^+(\omega\delta - e_2)\beta]] = \mu[\beta^+\beta][\delta\beta] - [\beta^+\beta]\}\rangle.
\end{aligned}$$

Let $\gamma := [\delta\alpha] - \mu[\delta\beta]$. We have

$$2[\delta\alpha] = e_1 + [\delta\alpha] - \mu[\delta\beta] = e_1 + \gamma \text{ and } 2\mu[\delta\beta] = e_1 - ([\delta\alpha] - \mu[\delta\beta]) = e_1 - \gamma.$$

Renaming $[\beta^+\beta] =: \beta^+$, $[\alpha^+\alpha] =: \alpha^+$, dropping the bracket notation and rewriting the relations with $\gamma's$ gives the desired description.

□

Proof of Theorem 2.1.10: By Theorems 2.2.4, 2.5.2 and Example 2.2.2,

$$per(\Lambda)/\langle B\rangle \simeq \mathcal{A}^{tr}/\mathcal{B}^{tr} \simeq (\mathcal{A}/\mathcal{B})^{tr},$$

and by Proposition 2.5.5,

$$(\mathcal{A}/\mathcal{B})^{tr} \simeq (\mathcal{A}/\mathcal{B})^{tr}_{\circ} \simeq \mathcal{P}(\Lambda[\omega^{-1}])^{tr} \simeq per(\Lambda[\omega^{-1}])$$

We conclude by using $per(\Lambda)/\langle B\rangle \simeq per(\Lambda)/thick(B)$ and Lemma 2.6.2.

□

Proof of Theorem 2.1.11: Let $\mathcal{A}$ and $\mathcal{B}$ be as in the Setting 2.5.1. Theorem 3.1 of [Gye24] gives the recollement:

$$\mathcal{D}(\mathcal{A}/\mathcal{B}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}(\mathcal{A}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}(\mathcal{B})$$

First $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{P}(\Lambda)) \simeq \mathcal{D}(\Lambda)$. Then one can show that $\mathrm{End}_{\mathcal{P}(\Lambda)^{pre-tr}}(B)$ is a formal DG algebra, and thus $\mathcal{D}(\mathcal{B}) \simeq \mathcal{D}(\mathrm{End}_{\mathcal{P}(\Lambda)^{pre-tr}}(B)) \simeq \mathcal{D}(H^*\mathrm{End}_{\mathcal{P}(\Lambda)^{pre-tr}}(B)) \simeq \mathcal{D}(K[x]/(x^2))$, by Lemma 2.5.6. Moreover $\mathcal{D}(\mathcal{A}/\mathcal{B}) \simeq \mathcal{D}((\mathcal{A}/\mathcal{B})_{\circ}) \simeq \mathcal{D}(\Lambda[\omega^{-1}])$, where the last equivalence is given by Proposition 2.5.5. We conclude by using Lemma 2.6.2.

□

Chapter 3

Formal Generators for A_∞-quotients of topological Fukaya categories

3.1 Introduction

3.1.1 A_∞-quotient and simple closed curve contraction

In [Jef22], the author considers families $f : X \to \mathbb{C}$ of symplectic manifolds with a singular fiber over 0, and defines the wrapped Fukaya category of the singular fiber to be a certain localization of the Fukaya category of a nearby fiber. Our work can be seen as an algebraic analogous construction in which we seek an explicit description of the localization.

Let $\hat{S}$ be a smooth graded marked surface and $\pi : \hat{S} \to S$ be the contraction of a simple closed curve γ on $\hat{S}$ of winding number zero. Let $\mathcal{F}(\hat{S})$ be the topological Fukaya category of $\hat{S}$ as defined in [HKK17], and let B be the object in this category corresponding to γ. We study the A_∞-category $\mathcal{F}(S) := \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$, where $\mathcal{B}$ is the full subcategory of $\mathcal{F}(\hat{S})$ on objects which become isomorphic to some elements in thick(B) when passing to the zero homology, and where $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$ denotes the A_∞-quotient category introduced in [LO06]. The goal of this chapter is to give a simpler definition for $\mathcal{F}(S)$ which remains Morita equivalent to $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$. The construction is summarized in the following diagram:

$$
\begin{array}{ccccc}
H^*\mathcal{A} & \longrightarrow & \mathcal{D}(H^*\mathcal{A}|B) & \xleftarrow{\sim} & \mathcal{F}_A(S) = H^*\mathcal{D} \\
\wr\downarrow & & \wr\downarrow & & \\
\mathcal{A} & \longrightarrow & \mathcal{D}(\mathcal{A}|B) & & \wr\downarrow \\
\downarrow & & \wr\downarrow & & \\
\mathcal{F}(\hat{S}) & \longrightarrow & \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B}) & \xleftarrow{\sim} & \mathcal{F}(S) = Tw\,\mathcal{F}_A(S) \\
 & & & & \\
\hat{S} & & \longrightarrow & & S
\end{array}
$$

Let A be an admissible dissection on S as defined in Definition 3.3.1. By subsection 3.3.4, it lifts to a collection of non-intersecting arcs on $\hat{S}$, which we also denote by A. It is then completed into a full formal arc system $\hat{A}$ on $\hat{S}$. As shown in [HKK17], $Tw\mathcal{F}_{\hat{A}}(\hat{S})$ is Morita equivalent to $\mathcal{F}(\hat{S})$. Thus the object B can be expressed as a twisted complex supported on the objects of $\hat{A}$ (see Subsection 3.3.5).

Let $\mathcal{A}$ be the full subcategory of $Tw\mathcal{F}_{\hat{A}}(\hat{S})$ supported on $A \sqcup \{B\}$, where A is seen as a subset of $\hat{A}$. It is shown in Lemma 3.5.2 (and the proof of Theorem 3.5.3) that $\mathcal{D}(\mathcal{A}|B)$ generates $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$. Moreover, taking a minimal model $H^*\mathcal{A}$ of the DG category $\mathcal{A}$ (Proposition 3.4.8) gives a Morita equivalence $\mathcal{D}(H^*\mathcal{A}|B) \to \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$. In fact, since B becomes isomorphic to zero in homology, we can restrict ourselves to the full subcategory $\mathcal{D}$ of $\mathcal{D}(H^*\mathcal{A}|B)$ supported on A, to obtain a Morita equivalence $\mathcal{D} \to \mathcal{D}(H^*\mathcal{A}|B) \to \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$. We show in Proposition 3.5.10 that $\mathcal{D}$ is a formal A_∞-category and give in Theorem 3.5.11 a description of the homology category $H^*\mathcal{D}$ as the path category of a quiver with relations. This motivates the definition $\mathcal{F}_A(S) := H^*\mathcal{D}$, which then satisfies the following theorem.

Theorem 3.1.1 *(Theorem 3.5.3) Following notations of Setting 3.4.1, there is a Morita equivalence:*

$$\mathcal{F}_A(S) \to \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B}).$$

By definition, the topological Fukaya category of S is taken as $\mathcal{F}(S) := Tw\mathcal{F}_A(S)$.

3.2 Notations for A_∞-categories

We first introduce the relevant comcepts and notations on A_∞-categories that will be useful for the rest of this chapter. We mainly follow [AP24], [HKK17], and [Sei08]. We fix a field K, that we suppose of characteristic zero. This hypothesis is used for example in Notation 3.5.6.

3.2.0.1 A_∞-categories

Recall that an A_∞-category $\mathcal{A}$ is *strictly unital* if each object X admits a *unit*, that is, a morphim e in $\mathcal{A}^0(X, X)$ such that:

- $\mu^2(a, e) = a$ and $\mu^2(e, a) = (-1)^{|a|}a$ for any post-composable / pre-composable homogeneous morphism a,
- $\mu^n(\dots, e, \dots) = 0$ for $n \neq 2$.

We refer to [Sei08](I.4) for the idempotent completion $\mathcal{A}^\natural$ of an A_∞-category. It is an enhancement of the idempotent completion for regular categories in the sense that $H^0(\mathcal{A}^\natural)$ is equivalent to $H^0(\mathcal{A})^\natural$ [Sei08](4c).

A strictly unital functor $F : \mathcal{A} \to \mathcal{B}$ is a *quasi-equivalence* of A_∞-categories if the induced functor H^*F is an equivalence of categories. In this case, the induced functor TwF between the A_∞-categories of twisted complexes $Tw\mathcal{A}$ and $Tw\mathcal{B}$, is also a quasi-equivalence. See for instance [Sei08] (Lemma 3.25).

More generally, an A_∞-functor F for which the induced functor $Tw(F)^\natural$ is a quasi-equivalence is called a *Morita equivalence*. The category $\mathcal{A}^{tr} := H^0Tw\mathcal{A}$ admits canonically a structure of triangulated category, and for a strictly unital functor $F : \mathcal{A} \to \mathcal{B}$, the induced functor F^{tr} is triangulated [Sei08](Lemma 3.30). Moreover, the idempotent completion $(\mathcal{A}^{tr})^\natural$ also admits a canonical triangulated structure [BS01]. Seing morphisms in $H^*\mathcal{A}$ as morphisms in degree zero between shifts of objects, we see that applying H^* to $Tw(F)^\natural$ induces an equivalence if and only if applying H^0 does, ie. F is a Morita equivalence if and only if $(F^{tr})^\natural$ is an equivalence.

3.2.0.2 Twisted complexes

Given an A_∞-category $\mathcal{A}$, we define the additive enlargement $add\mathbb{Z}\mathcal{A}$ and the A_∞-category of twisted complexes $Tw\mathcal{A}$ as in [Sei08](Section 3). Recall that the involved higher multiplications are given by the following formulas:

$$\begin{aligned} \mu^n_{\mathrm{add}\mathbb{Z}\mathcal{A}}(\phi_n \otimes a_n, ..., \phi_1 \otimes a_1) &= (-1)^{\sum_{i<j} |\phi_i|\|a_j\|}\phi_n ... \phi_1 \otimes \mu^n_{\mathcal{A}}(a_n, ..., a_1), \\ \mu^n_{Tw\mathcal{A}}(a_n, ..., a_1) &= \sum \mu_{add\mathbb{Z}\mathcal{A}}(\delta_n, ..., \delta_n, a_n, \delta_{n-1}, ..., \delta_1, a_1, \delta_0, ..., \delta_0). \end{aligned} \tag{3.1}$$

For X in $\mathcal{A}$ and $k \in \mathbb{Z}$, we introduce the notation $X[k] := K[k] \otimes X \in \mathrm{add}\mathbb{Z}\mathcal{A}$. A morphism from $X[k]$ to $Y[l]$ is of the form $s^{l-k} \otimes a$, where s^{l-k} denotes the identity morphism between $K[k]$ and $K[l]$. We have $|s^k| = -k$. When there is no ambiguity, we will drop the s and write $s^{l-k} \otimes a$ simply as a.

By definition, the differential δ of a twisted complex is a linear combination of elements of the form $s^l \otimes d$ of degree one. Thus $|d| = l+1$. If $\mu^n_{\mathcal{A}} = 0$ for all $n \geq 3$, then $\mu^n_{Tw\mathcal{A}}$ also vanishes for $n \geq 3$. In this case we have:

$$\begin{aligned} \mu^1_{Tw\mathcal{A}}(s^{l-k} \otimes a) &= \mu^1_{\mathrm{add}\mathbb{Z}\mathcal{A}}(s^{l-k} \otimes a) + \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{A}}(\delta', s^{l-k} \otimes a) + \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{A}}(s^{l-k} \otimes a, \delta) \\ &= s^{l-k} \otimes \mu^1_{\mathcal{A}}(a) + \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{A}}(\delta', s^{l-k} \otimes a) + \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{A}}(s^{l-k} \otimes a, \delta) \\ &= s^{l-k} \otimes \mu^1_{\mathcal{A}}(a) + \sum_i \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{A}}(s^{k_i-l} \otimes d'_i, s^{l-k} \otimes a) + \sum_j \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{A}}(s^{l-k} \otimes a, s^{k-l_j} \otimes d_j) \\ &= s^{l-k} \otimes \mu^1_{\mathcal{A}}(a) + \sum_i (-1)^{(k-l)\|d'_i\|} s^{k_i-k} \otimes \mu^2_{\mathcal{A}}(d'_i, a) + \sum_j (-1)^{(l_j-k)\|a\|} s^{l-l_j} \otimes \mu^2_{\mathcal{A}}(a, d_j), \end{aligned} \tag{3.2}$$

where the $s^{k_i-l} \otimes d_i$ are the components of δ starting at the codomain of $s^{l-k} \otimes a$, and symmetricaly for the $s^{k-l_j} \otimes d_j$.

More generally, and without vanishing assumption, one has for a chain of morphisms passing through $X_0[k_0], X_1[k_1], ...$ to $X_n[k_n]$:

$$\begin{aligned} \mu^n_{\mathrm{add}\mathbb{Z}\mathcal{A}}(s^{k_n-k_{n-1}} \otimes a_n, ..., s^{k_1-k_0} \otimes a_1) &= (-1)^{\sum_{i<j}(k_{i-1}-k_i)\|a_j\|} s^{k_n-k_0} \otimes \mu^n_{\mathcal{A}}(a_n, ..., a_1) \\ &= (-1)^{\sum_{1<j}(k_0-k_{j-1})\|a_j\|} s^{k_n-k_0} \otimes \mu^n_{\mathcal{A}}(a_n, ..., a_1). \end{aligned} \tag{3.3}$$

3.3 Marked surfaces with conical singularities

The definition of the topological Fukaya category $(\mathcal{F}(\hat{S}), \hat{\mu})$ of a smooth graded marked surfaces $\hat{S}$ is given in [HKK17]. Bases of the morphism spaces are given by boundary paths, and the multiplication

$\hat{\mu}^2$ is given by the concatenation of paths: $\hat{\mu}^2(b,a) = (-1)^{|a|}ba$, where $|a|$ the degree of a induced by the grading. By convention, the concatenation of two paths with non compatible endpoints/starting points is zero.

3.3.1 Admissible dissections on marked surfaces with conical singularities

Let $(\hat{S}, M, \eta)$ be a smooth graded marked surface and let $\{\gamma_l\}$ be a collection of pairwise disjoint non-isotopic simple closed curves on $\hat{S}$, of winding number zero. Let S be the topological space obtained by contracting each γ_l to a point, and let $\pi : \hat{S} \to S$ be the corresponding quotient map. Let C be the set of singular points of S. The tuple (S, M, C) is a *marked surface with conical singularities*.

Recall that an arc on (S, M, C) is a locally embedded closed interval in S which intersects transversely M at its endpoints, and which is not isotopic to an interval in M by an isotopy keeping its endpoints in M. This condition will always be assumed when referring to an isotopy of arc. A decomposition of an arc γ induced by C is a collection of arcs $(\gamma_i)_{1\leq i\leq r}$ called components, such that γ is the concatenation $\gamma_r \dots \gamma_1$, and each γ_i has its endpoints in $M \sqcup C$, and does not intersect C otherwise. A collection of arcs A on (S, M, C) is said to be in minimal position if the number of intersections between elements of A is minimal, as well as the number of intersections between elements of A and of C. An *arc system* on (S, M, C) is a collection of arcs in minimal position, such that their decomposition induced by C have non-isotopic components, and such that they do not intersect in $S \backslash C$.

Let A be an arc system on S such that each element of A intersects C at most once. For $c \in C$, let A_c be the set of arcs of A passing through c, and let $A_s = A \backslash \sqcup_{c\in C} A_c$. The orientation of S induces a cyclic order on A_c on each side of the singularity c, encoded in two cyclic permutations σ_+, σ_-. The arcs in A_c are said to be *cyclically compatible* if $\sigma_- = \sigma_+^{-1}$. Figure 3.3.1 gives an example of cyclically compatible arcs at a singularity, with cyclic permutation (123).

Definition 3.3.1 *An arc system A on a marked surface with conical singularities is an admissible dissection if the following conditions hold:*

- *Each element of A intersects C at most once,*
- *For each $c \in C$, the arcs in A_c are cyclically compatible,*
- *The arcs of A cut S into polygons, each containing exactly one unmarked boundary segment.*

A grading *G on a admissible dissection A is the data of an integer $g(\alpha)$ for each minimal angle α between arcs of A sharing an endpoint in M.*

When cutting along an arc of A_c, we consider that it splits the singularity c. The points of C can therefore be vertices of the obtained polygons. When choosing a grading, an interger is assigned only at minimal angle form by arcs sharing an endpoint in M, and not at minimal angles that arcs might form by intersect at a singularity $c \in C$.

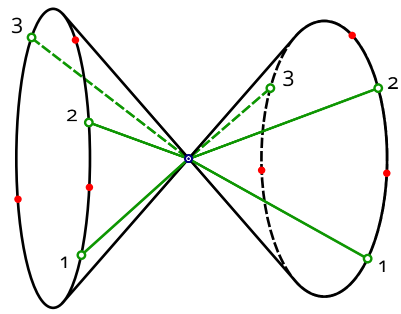


Figure 3.3.1: A marked surface with conical singularities and admissible dissection.

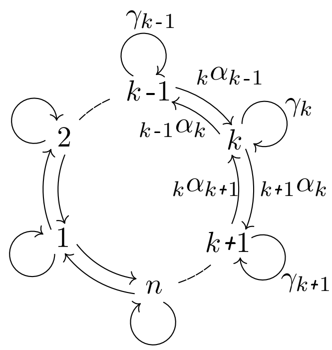
3.3.2 Pinched gentle algebras

In this subsection we introduce a class of graded algebras which will appear as Ext-algebra of formal generators for A_∞-quotients of topological Fukaya categories.

Given a vertex k of a quiver, let $S(k)$ be the *star* of k, that is, the set of arrows for which k is a source or target. If k is a vertex of a gentle quiver (Q, I), there are arrows $\alpha^-, \alpha^+, \beta^-, \beta^+ \in Q_1$ satisfying $\tau(\alpha^-) = \sigma(\alpha^+) = k$ and $\tau(\beta^-) = \sigma(\beta^+) = k$, and $\beta^+\alpha^-, \alpha^+\beta^- \in I$, provided that we allow some identifications between $\{\alpha^-, \beta^-\}$ and $\{\alpha^+, \beta^+\}$, and that some of these arrows might be zero. See the left-hand side of Figure 3.3.2 for an illustration of the generic case. An *ordering* of the star $S(k)$ in (Q, I) is a choice of sets $S_1(k)$ and $S_2(k)$ given either by $(S_1(k), S_2(k)) = (\{\alpha^-, \alpha^+\}, \{\beta^-, \beta^+\})$ or $(S_1(k), S_2(k)) = (\{\beta^-, \beta^+\}, \{\alpha^-, \alpha^+\})$.

Definition 3.3.2 *A graded pinched gentle algebra is a graded K-algebra Λ isomorphic to a quotient $(KQ/\langle I\rangle, |\,.\,|)$ with $Q_1 = Q_1^g \sqcup A_1 \sqcup \ldots \sqcup A_r$ and $I = I^g \sqcup I_1 \sqcup \ldots \sqcup I_r$, satisfying:*

- *$(Q^g := (Q_0, Q_1^g), I^g)$ is a gentle bound quiver, which might be disconnected,*
- *The arrows of A_i have degree zero and form a subquiver of the form:*



- *The vertex supports C_i of the set of arrows A_i are disjoint,*
- *The relations in I_i are of the form*

$$\beta^+(\gamma_k + e_k),\ (\gamma_k + e_k)\beta^-,\ \alpha^+(\gamma_k - e_k),\ (\gamma_k - e_k)\alpha^-,$$
$$_{k+1}\alpha_k\alpha^-,\ _{k-1}\alpha_k\alpha^-,\ \alpha^+{}_k\alpha_{k+1},\ \alpha^+{}_k\alpha_{k-1},$$
$$_{k+1}\alpha_k\beta^-,\ _{k-1}\alpha_k\beta^-,\ \beta^+{}_k\alpha_{k+1},\ \beta^+{}_k\alpha_{k-1},$$

for all vertex $k \in C_i$, where $\{\alpha^-, \alpha^+\}$ and $\{\beta^-, \beta^+\}$ come from the choice of an ordering of $S(k)$ in (Q^g, I^g). The other relations are:

$$
\begin{aligned}
{}_k\alpha_i \gamma_i &= \gamma_k {}_k\alpha_i \\
{}_i\alpha_k {}_k\alpha_i &= \gamma_i^2 - e_i \\
{}_k\alpha_i^+ (\gamma_i - e_i)^{l^-} &= {}_k\alpha_i^- (\gamma_i + e_i)^{l^+}
\end{aligned}
$$

for all $i \neq k \in \{1, \ldots, n\}$, where ${}_k\alpha_i^+ = {}_k\alpha_{k-1} \ldots {}_{i+1}\alpha_i$ and ${}_k\alpha_i^- = {}_k\alpha_{k+1} \ldots {}_{i-1}\alpha_i$ (with each term ${}_{l\pm1}\alpha_l$ apearing at most once), and l^+ is the length of ${}_k\alpha_i^+$ minus one, l^- the length of ${}_k\alpha_i^-$ minus one.

We introduce the notation $Q_1^p = A_1 \sqcup \ldots \sqcup A_r$ and $I^p = I_1 \sqcup \ldots \sqcup I_r$. The underlying gentle algebra of Λ *is $\Lambda^g := KQ^g/\langle I^g \rangle$.*

Example 3.3.3

(1) The right-hand side of Figure 3.3.2 shows a pinched gentle algebra with $r = 1$ and $|C_1| = 1$. In this case, A_1 contains a single loop.

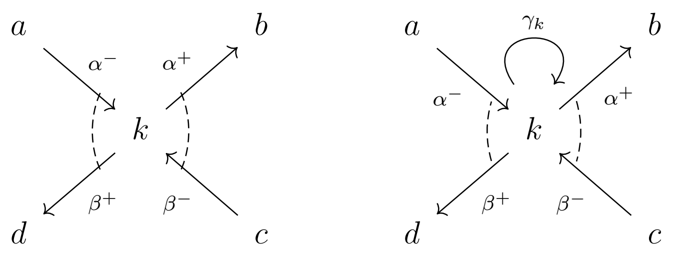


Figure 3.3.2: A pinched gentle algebra with special loop γ_k (right), and its underlying gentle algebra (left).

(2) The left-hand side of Figure 3.3.3 shows the case $Q_0 = \{1, 2\}$, $Q_1^g = \emptyset$, $r = 1$ and $|C_1| = 2$, and the right-hand side shows the case $Q_0 = \{1, 2, 3\}$, $Q_1^g = \emptyset$, $r = 1$ and $|C_1| = 3$.

(3) The case where $Q_0 = \{0, 1, 2\}$, Q_1^g is the oriented 3-cycle, $r = 1$ and $C_1 = \{1, 2\}$ is represented in the lower right-hand corner of Figure 3.3.5.

Remark 3.3.4

1. *By convention, when C_j contains a single vertex k, A_j contains only the loop γ_k. Moreover, when C_j contains two elements, the arrows ${}_{k+1}\alpha_k$ and ${}_{k-1}\alpha_k$ coincides for each $k \in C_j$.*
2. *A subclass of these algebras already appeared under the same name in [Bod25], namely when all C_j contain a single vertex. The above definition is thus a generalization.*

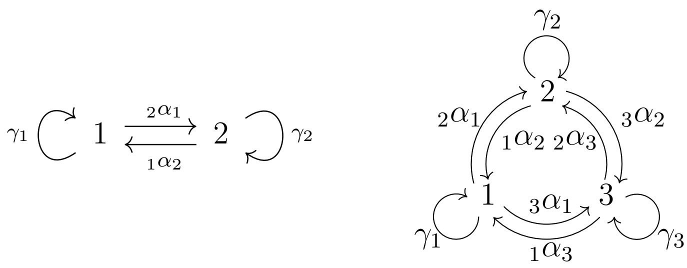


Figure 3.3.3: Pinched cycles of size 2 and 3.

3. *One can use the last three relations of the definition to deduce the following ones:*

$$
\begin{aligned}
{}_i\alpha_i^+ &= (\gamma_i - e_i)(\gamma_i + e_i)^{n-1} \\
{}_i\alpha_i^- &= (\gamma_i + e_i)(\gamma_i - e_i)^{n-1}
\end{aligned}
$$

4. *Let Λ' be the pinched gentle algebra obtained from Λ by reversing the order of a cycle C_j, identified with $\{1, \ldots, n\}$ and its natural order, and by changing the choice of ordering of the stars $S(k)$ for $k \in C$. Let ${}_{k\pm1}\alpha_k, \gamma_k$ be the arrows of $A_j \subset Q_1^p$ in Λ, and let ${}_{k\pm1}\alpha'_k, \gamma'_k$ the corresponding one in Λ'. Then the map which sends ${}_{k\pm1}\alpha_k, \gamma_k$ to $-{}_{k\pm1}\alpha'_k, -\gamma'_k$ and is the identity otherwise is an isomorphism between Λ and Λ'.*

5. *The relations in I^p are such that for all vertices k and l in Q_0 that do not belong to a same cycle C_j, one has an isomorphism of vector spaces $e_l\Lambda e_k \simeq e_l\Lambda^g e_k$.*

6. *If a cycle C_l contains a single element k, and the corresponding arrows α^- and α^+ (resp. β^- and β^+) are both zero, then one can replace γ_k by $\gamma_k + e_k$ (resp. $\gamma_k - e_k$) to obtain gentle relations, and thus decrease the number of cycles C_j by one.*

An explicit K-basis for pinched gentle algebras is given in Subsection 3.5.3, as well as in Remark 3.5.14.

3.3.3 The quiver with relations of an admissible dissection

We now associated to a marked surface with conical singularities (S, M, C) endowed with an admissible dissection A a pinched gentle bounded quiver (Q, I). As usual, we suppose that A is in minimal position on S.

The set of vertices Q_0 coincides with the set of arcs A. The gentle bound quiver (Q^g, I^g) is constructed as in the smooth case. Namely, the set of arrows Q_1^g is the set of boundary paths which start and end at arcs of A, but does not cross any other arcs. The ideal I^g is generated by the quadratic relations that come from composable arrows which do not correspond to composable paths.

The set of vertices $C_1, \ldots, C_r \subseteq Q_0$ are given by the sets A_c of arcs of A passing through c, for each singular point $c \in C$. One chooses an order on C_j corresponding to $c \in C$, as well as an ordering of its stars, in the following way. Split the singularity c in two and choose one side. The order on C_j

is the one obtained by turning around c following the orientation of the surface. The ordering of the stars $S(k)$, for vertices k in C, are such that all $S_1(k)$ lie on the side of the singularity that we chose.

Example 3.3.5

(1) Figure 3.3.4 depicts the marked surface of Figure 3.3.1 with its conical singularity split. The pinched gentle quiver is drawn on the singular surface, with a loop γ_k at each vertex k omitted. The complete quiver is depicted in the right-hand side of Figure 3.3.3. In this example the left side of the singularity was choosen and the orientation of the surface is drawn clockwise. By Remark 3.3.4.(4), the choice of a different side of the singularity gives an isomorphic pinched gentle algebra.

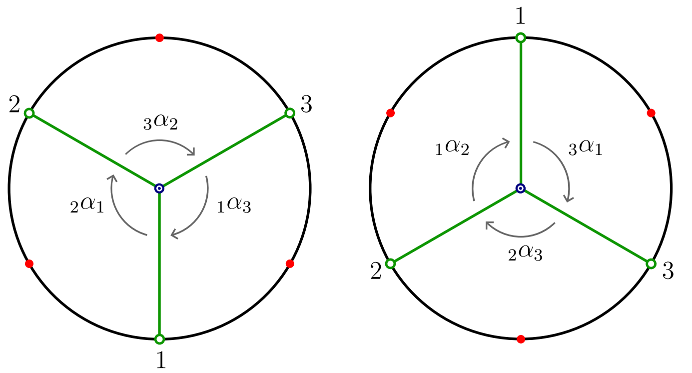


Figure 3.3.4: A pinched cylinder with split conical singularity, endowed with an admissible dissection and its associated quiver.

(2) The top of Figure 3.3.5 shows a marked surface with conical singularity obtained by the contraction of a simple closed curve on a torus with one boundary component (see Figure 3.3.7). The same marked surface is represented with its conical singularity split in the lower left-hand corner. They both are endowed with the same admissible dissection and the corresponding quiver is drawn in the lower right-hand corner.

Definition 3.3.6 *Let (S, M, C) be a marked surface with conical singularities, and let A be an admissible dissection on S. The* $\mathbb{Z}$-graded category of A *is defined by $\mathcal{F}_A(S) = \mathcal{P}(Q, I)$, where (Q, I) is the pinched gentle bounded quiver associated to A as in the previous paragraph, and $\mathcal{P}(Q, I)$ is the path category of (Q, I), whose objects are Q_0 and morphism spaces are $\mathcal{P}(Q, I)(i, k) = e_k \Lambda e_i$ where $\Lambda = KQ/\langle I \rangle$.*

Remark 3.3.7 *One can also generalize the construction of [OPS25] for gentle algebras, in order to associate to a graded pinched gentle quiver a marked surface with conical singularities endowed with a graded admissible dissection.*

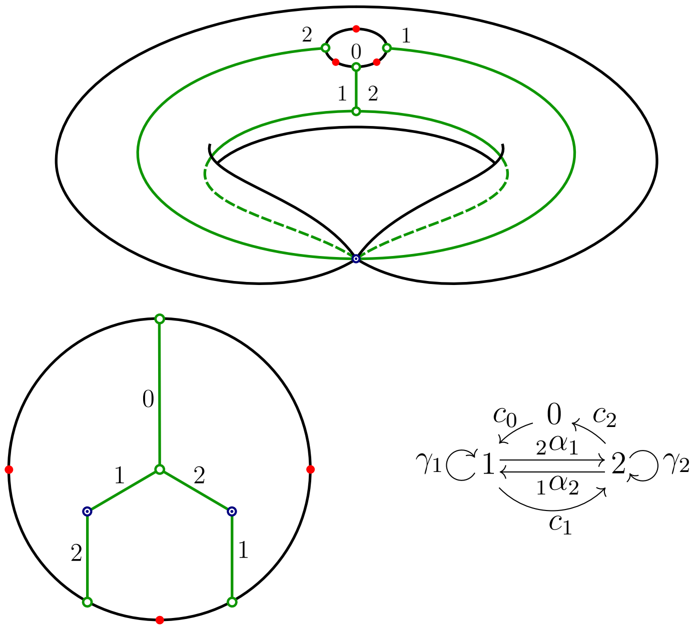


Figure 3.3.5: A pinched torus (top) depicted with split conical singularity (bottom left) and its associated quiver (bottom right).

3.3.4 Lifting of admissible dissections

Let S be a marked surface with conical singularities obtained by a contraction $\pi : \hat{S} \to S$ of a collection of simple closed curves on a smooth marked surface $\hat{S}$. We lift each admissible dissection A on S to an admissible dissection $\hat{A}$ (of marked surface without conical singularities) on $\hat{S}$. See Figure 3.3.6 for an illustration.

First, note that the restriction $\pi : \hat{S}\backslash\{\gamma_l\} \to S\backslash C$ is a homeomorphism. This enables us to lift each arc of A_s to an arc on $\hat{S}$. For $c \in C$, we label the elements of A_c with $\{1, \dots, n\}$ in such a way that it is compatible with the cyclic ordering. Up to homotopy, there is $\mathbb{Z}$ choices of lift for an element of A_c. Since the elements of A_c are cyclically compatible, the choice of a lift for the arc 1 imposes a unique lift for the arcs $\{2, \dots, n\}$ in order to obtain a collection of non-intersecting arcs on $\hat{S}$. Doing this for each $c \in C$, we obtain a collection of arcs on $\hat{S}$, which we also call A. We also still use the decomposition of A into A_s and the sets A_c when refering to the lifted arcs. By construction, elements of A are pairwise disjoint non-isotopic arcs, and they cut $\hat{S}$ into polygons. The only polygon containing more than one unmarked boundary segment are the one cut by two consecutive arcs of A_c, for each $c \in C$. We can add arcs to A in the following way in order to obtain an admissible dissection $\hat{A}$.

Let $c \in C$ be a singularity corresponding to a simple closed curve γ. Let $i, i+1$ be two consecutive arcs in A_c (the label being taken modulo n), and let γ^i be a portion of γ that has an endpoint p in i and an endpoint q in $i+1$, and that does not intersect the arcs of A_c otherwise. Let $i = \delta'\delta$ be a decomposition of i such that δ and δ' go from an extremity of i to p, let $i+1 = \rho'\rho$ be a similar decomposition for $i+1$, and choose them in such a way that following the reverse orientation of the boundary induces a morphism a_i from i to the concatenation $\eta = \rho'\gamma^i\delta$, and a morphism b_{i+1} from $i+1$ to η. We label η by i'. Adding all these arcs to A gives an admissible dissection $\hat{A}$ of $\hat{S}$.

If G is a grading on A, then define $\hat{G}$ by $G(a_i) = G(b_i) = 0$ for all $c \in C$ and $i \in A_c$, and by $\hat{g}(\alpha) = g(\alpha)$ for all minimal angle α between arcs of A. By construction, each simple closed curve γ_l has winding number zero. Let $\mathcal{F} := \mathcal{F}_A(\hat{S})$ be the minimal A_∞-category of $\hat{A}$. Since $\hat{A}$ is an admissible dissection, its only non-trivial multiplication is $\hat{\mu}^2$. For all i, the degree of a_i and b_i is zero.

Example 3.3.8

(1) *Figure 3.3.6 shows a lifting of the dissection of a pinched cylinder containing three arcs passing through the singularity, as depicted in Figure 3.3.4.*

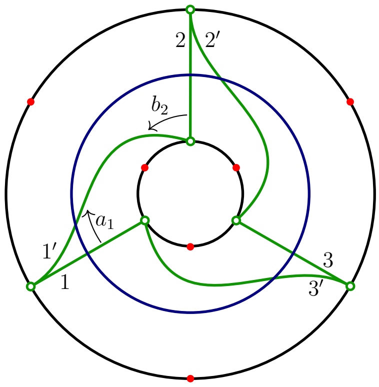


Figure 3.3.6: A smooth marked surface with a lifted admissible dissection.

(2) *The top of Figure 3.3.7 shows a smooth marked surface with a simple closed curve (blue) whose contraction gives the marked surface with conical singularity of Figure 3.3.5. The same marked surface is represented as a square with opposite sides identified, in the lower left-hand corner. They both are endowed with the admissible dissection obtained by lifting the one of Figure 3.3.5. The corresponding quiver is drawn in the lower right-hand corner.*

3.3.5 From conical singularities to band objects

Let c be in C and let γ be the corresponding simple closed curve on $\hat{S}$. Choosing an arbitrary grading for γ, as well as parameters $(m, \lambda) \in \mathbb{N}^* \times K$, allows us to construct a twisted complex (B, δ) of $Tw\mathcal{F}$. For $(m, \lambda) = (1, 1)$, and for the appropriate grading, section (4.1) of [HKK17] shows that B can be expressed as

$$B = \bigoplus_{i=1}^{n} i[1] \oplus \bigoplus_{i=1}^{n} i'.$$

The non-zero terms of the differential $\delta = (\delta_{i,j})_{i,j}$ are $\delta_{i',i} = a_i$ and $\delta_{(i-1)',i} = b_i$. As always, the indices i are taken modulo n.

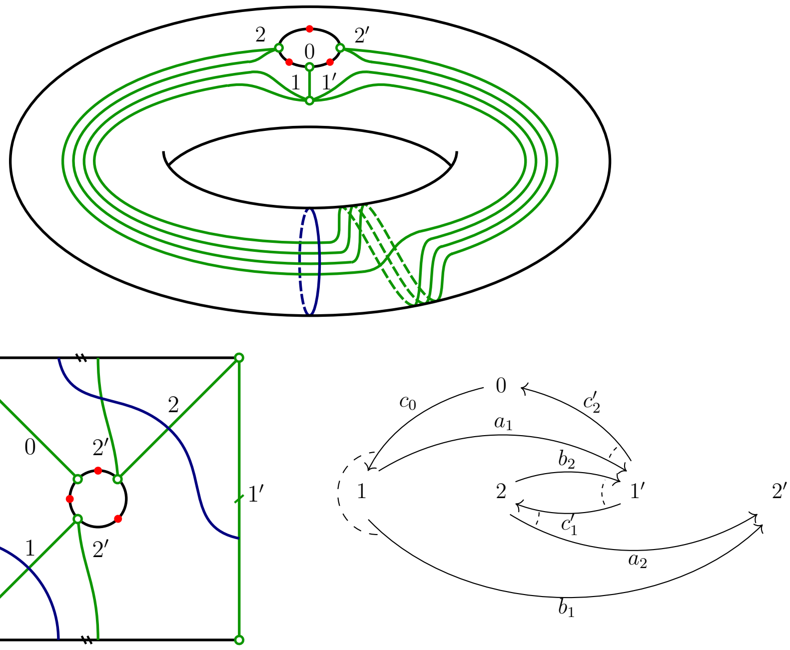
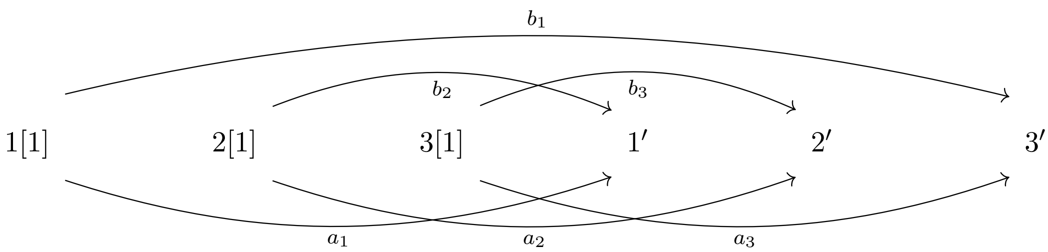


Figure 3.3.7: A lifted admissible dissection on a torus (top) depicted as a square with opposite sides identified (bottom left) and its associated quiver (bottom right).

Example 3.3.9 *The twisted complex corresponding to the blue simple closed curve of Figure 3.3.6, with parameters $(m, \lambda) = (1, 1)$, is given by:*



3.4 Minimal model for a generator of the topological Fukaya category

3.4.1 Setting

In this subsection we give a basis and describe the differential of the DG category $\mathcal{A}$ introduced in the following setting.

Setting 3.4.1

- *Let $\hat{S}$ be a smooth marked surface and γ a simple closed curve on $\hat{S}$ of winding number zero,*

- *Let S be the marked surface with one conical singularity c obtained by the contraction $\pi : \hat{S} \to S$ of γ in $\hat{S}$,*
- *Let $\mathcal{F}_A(S)$ be the $\mathbb{Z}$-graded category of A defined in Definition 3.3.6,*
- *Let A be an admissible dissection on S (Definition 3.3.1). The set A_c of arcs of A passing through c is identified with the ordered set $\{1, \dots, n\}$, compatibly with the cyclic ordering of the arcs at c,*
- *Let $\hat{A}$ be the admissible dissection of $\hat{S}$ obtained by lifting A (see Subsection 3.3.4). Elements of A are labelled in the same way both on S and $\hat{S}$. The set $\hat{A}$ is the disjoint union of*
  - *$A = A_c \sqcup A_s$, where A_s is the set of arcs of A that do not intersect c,*
  - *$A'_c = \{1', \dots, n'\}$, the set of arcs that are added to A in order to obtain an admissible dissection on $\hat{S}$,*
- *Let $\mathcal{F} := \mathcal{F}_{\hat{A}}(\hat{S})$ be the minimal A_∞-category of the arc collection $\hat{A}$. Its only non-trivial multiplication is $\hat{\mu}^2$. For all i, there are morphisms $a_i \in \mathcal{F}(i, i')$ and $b_i \in \mathcal{F}(i, (i-1)')$ of degree zero,*
- *Let B be the twisted complex $\bigoplus_{i=1}^n i[1] \oplus \bigoplus_{i=1}^n i'$ of $\mathcal{F}(S) = Tw\mathcal{F}$, whose non zero terms of the differential are $\delta_{i',i} = a_i$ and $\delta_{(i-1)',i} = b_i$. The A_∞-multiplications of $Tw\mathcal{F}$ are denoted by $\tilde{\mu}$,*
- *Let $\mathcal{B}$ be the full subcategory of $\mathcal{F}(S)$ supported on objects which are isomorphic to elements in $thick(B)$ after passing to the zero homology,*
- *Let $\mathcal{A}$ be the full subcategory of $Tw\mathcal{F}$ supported on A and B.*

Even though the category $\mathcal{A}$ does not generate $Tw\mathcal{F}$, we will see in Lemma 3.5.2 that it induces a generator $\mathcal{D}(\mathcal{A}|B)$ of the A_∞-quotient $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$.

3.4.1.1 The canonical basis of $\mathcal{A}$

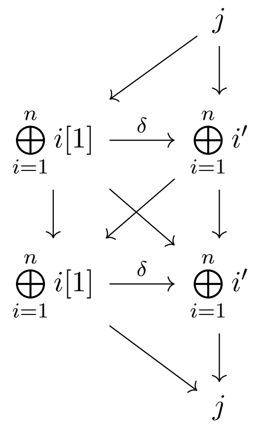
As mentioned before, for $i, j \in A$, a basis of $\mathcal{A}(i, j) = \mathcal{F}(i, j)$ is given by the boundary paths from i to j.

Note that by construction of the lifted dissection $\hat{A}$, for $i \in A_c$ and $j \in A$, each non trivial boundary path ρ from j to i' factors as $\rho = a_i\rho'$ or $\rho = b_{i+1}\rho'$. Similarly, each non trivial boundary path ρ from i to j factors as $\rho = \rho' a_i$ or $\rho = \rho' b_i$.

Let $j \in A$. The definition of the morphism spaces between twisted complexes allows for the following identifications:

$$
\begin{aligned}
&- \mathcal{A}(j, B) = \bigoplus_{i=1}^n \mathcal{F}(j, i)[-1] \oplus \bigoplus_{i=1}^n \mathcal{F}(j, i'),\\
&- \mathcal{A}(B, B) = \bigoplus_{k=1}^n \bigoplus_{l=1}^n (\mathcal{F}(k, l) \oplus \mathcal{F}(k, l')[1] \oplus \mathcal{F}(k', l)[-1] \oplus \mathcal{F}(k', l')),\\
&- \mathcal{A}(B, j) = \bigoplus_{i=1}^n \mathcal{F}(i, j)[1] \oplus \bigoplus_{i=1}^n \mathcal{F}(i', j).
\end{aligned}
\tag{3.4}
$$

The situation is summarized in the following diagram:



The fact that a boundary path ρ from i to j represents a morphism between i and j, or i and B, etc., will be deduced from the context. Recall that $|\_|$ denotes the grading of $\mathcal{F}$. The degrees in $\mathcal{A}$ are :

- In $\mathcal{A}(j,B)$: $\rho \in \mathcal{F}(j,i)$ has degree $|\rho|_{\mathcal{A}} = |\rho| - 1$, and $\rho \in \mathcal{F}(j,i')$ has degree $|\rho|_{\mathcal{A}} = |\rho|$,
- In $\mathcal{A}(B,j)$: $\rho \in \mathcal{F}(i,j)$ has degree $|\rho|_{\mathcal{A}} = |\rho| + 1$, and $\rho \in \mathcal{F}(i',j)$ has degree $|\rho|_{\mathcal{A}} = |\rho|$.
- In $\mathcal{A}(B,B)$:

  $\rho \in \mathcal{F}(k,l)$ has degree $|\rho|_{\mathcal{A}} = |\rho|$, $\rho \in \mathcal{F}(k,l')$ has degree $|\rho|_{\mathcal{A}} = |\rho| + 1$,
  $\rho \in \mathcal{F}(k',l)$ has degree $|\rho|_{\mathcal{A}} = |\rho| - 1$, and $\rho \in \mathcal{F}(k',l')$ has degree $|\rho|_{\mathcal{A}} = |\rho|$,

3.4.1.2 Differential of the DG category $\mathcal{A}$

Since $\mathcal{F}$ has no non-trivial $\hat{\mu}^n$ for $n \geq 3$, the same holds for $\mathcal{A}$. We now give a complete list of the non-vanishing $\tilde{\mu}^1$ using Equation 3.2.

The calculation is illustrated in the following examples. Let ρ be a path in $\mathcal{F}(j,i)$, seen as an element of $\mathcal{A}(j,B)$. It is of the form $s^1 \otimes \rho$ when we forget the simplified notation. We have:

$$\begin{aligned}\tilde{\mu}^1(s^1 \otimes \rho) &= \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{F}}(s^{-1} \otimes (a_i + b_i), s^1 \otimes \rho) \\ &= (-1)^{|s^1|\|a_i\|} s^0 \otimes \hat{\mu}^2(a_i,\rho) + (-1)^{|s^1|\|b_i\|} s^0 \otimes \hat{\mu}^2(b_i,\rho) \\ &= -s^0 \otimes ((-1)^{|\rho|} a_i\rho + (-1)^{|\rho|} b_i\rho).\end{aligned}$$

Dropping the s gives $\tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}(a_i\rho + b_i\rho)$. For $\rho \in \mathcal{F}(k,l)$ seen as an element of $\mathcal{A}(B,B)$:

$$\begin{aligned}\tilde{\mu}^1(\rho) = \tilde{\mu}^1(s^0 \otimes \rho) &= \mu^2_{\mathrm{add}\mathbb{Z}\mathcal{F}}(s^{-1} \otimes (a_l + b_l), s^0 \otimes \rho) \\ &= (-1)^{|s^0|\|a_l + b_l\|} s^{-1} \otimes \hat{\mu}^2(a_l + b_l, \rho) = (-1)^{|\rho|}(a_l\rho + b_l\rho).\end{aligned}$$

The other cases are computed similarly.

- In $\mathcal{A}(j,B)$: for $\rho \in \mathcal{F}(j,i)$, $\tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}(a_i\rho + b_i\rho)$,

- In $\mathcal{A}(B, j)$: for $\rho \in \mathcal{F}(i', j)$, $\tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}(\rho a_i + \rho b_{i+1})$.
- In $\mathcal{A}(B, B)$:
  - for $\rho \in \mathcal{F}(k, l)$, $\tilde{\mu}^1(\rho) = (-1)^{|\rho|}(a_l\rho + b_l\rho)$,
  - for $\rho \in \mathcal{F}(k', l)$, $\tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}(\rho a_k + \rho b_{k+1} + a_l\rho + b_l\rho)$,
  - for $\rho \in \mathcal{F}(k', l')$, $\tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}(\rho a_k + \rho b_{k+1})$.

3.4.2 Transfer of A_∞-structures

The following proposition is a special case of Proposition 1.12 of [Sei08] (see Remark 1.13 that follows). It is a generalisation of Theorem 1 of [Kad80].

Let $\mathcal{B}$ be an A_∞-category with higher multiplications $\tilde{\mu}^n_{\mathcal{B}}$. Suppose that for each pair (x, y) of objects of $\mathcal{B}$, a decomposition $\mathcal{B}(x, y) = H^*\mathcal{B}(x, y) \oplus C^*(x, y)$ is given, where $H^*\mathcal{B}(x, y)$ is the homology of $\mathcal{B}(x, y)$ seen as a complex with differential $\tilde{\mu}^1_{\mathcal{B}}$ and $C^*(x, y)$ is an acyclic complement.

Let $f^1 : H^*\mathcal{B}(x, y) \to \mathcal{B}(x, y)$ be the inclusion and $g^1 : \mathcal{B}(x, y) \to H^*\mathcal{B}(x, y)$ be the projection with respect to this decomposition. Let T^1 be an endomorphism (of graded vector space) of $\mathcal{B}(x, y)$ of degree -1 that vanishes on $H^*\mathcal{B}(x, y)$ and is a contracting homotopy for $C^*(x, y)$, that is, it satisfies:

$$\tilde{\mu}^1_{\mathcal{B}} T^1 + T^1 \tilde{\mu}^1_{\mathcal{B}} = f^1 g^1 - \mathrm{id}. \tag{3.5}$$

Proposition 3.4.2 *[Sei08](Prop-1.12)*

One can define a A_∞-category $\mathcal{A}$ with objects $Ob(\mathcal{A}) = Ob(\mathcal{B})$, morphism spaces $H^*\mathcal{B}(x, y)$, first order structure map $\mu^1 = 0$, as well as a quasi-isomorphism $f : \mathcal{A} \to \mathcal{B}$ which is the identity on objects and whose first order map is the inclusion f^1. The higher μ^n and f^n are given by the following recursive formulas:

$$\begin{aligned} f^n(x_n, \ldots, x_1) &= \sum_r \sum_{s_1, \ldots, s_r} T^1(\tilde{\mu}^r(f^{s_r}(x_n, \ldots, x_{n-s_r+1}), \ldots, f^{s_1}(x_{s_1}, \ldots, x_1))), \\ \mu^n(x_n, \ldots, x_1) &= \sum_r \sum_{s_1, \ldots, s_r} g^1(\tilde{\mu}^r(f^{s_r}(x_n, \ldots, x_{n-s_r+1}), \ldots, f^{s_1}(x_{s_1}, \ldots, x_1))), \end{aligned} \tag{3.6}$$

where the sums are taken over all partitions $n = s_1 + \ldots + s_r$ with $r \geq 2$.

3.4.3 Setup for the homotopy transfer

We now give a decomposition of the morphism spaces of $\mathcal{A}$ that will allow us to apply Proposition 3.4.2. Note that a different choice of decomposition would give a different description of the multiplication μ^n obtained by transfer.

Lemma 3.4.3 *For all pairs (x, y) of objects in $\mathcal{A}$, the following gives a decomposition of graded vector spaces*

$$\mathcal{A}(x, y) = H^*(x, y) \oplus I^*(x, y) \oplus C^*(x, y),$$

where $I^*(x, y)$ denotes the image of $\tilde{\mu}^1$, $H^*(x, y)$ is a complement of $I^*(x, y)$ in the kernel of $\tilde{\mu}^1$, and $C^*(x, y)$ is a complement of the kernel of $\tilde{\mu}^1$:

- $\forall i,j\in A$,

$$H^*(i,j)=\mathcal{F}(i,j),\ I^*(i,j)=0,\ C^*(i,j)=0,$$

- $\forall j\in A_s$,

$$H^*(j,B)=0,\ I^*(j,B)=\bigoplus_{i\in A_c}\mathcal{F}(j,i'),\ C^*(j,B)=\bigoplus_{i\in A_c}\mathcal{F}(j,i)[-1],$$

$$H^*(B,j)=0,\ I^*(B,j)=\bigoplus_{i\in A_c}\mathcal{F}(i,j)[1],\ C^*(B,j)=\bigoplus_{i\in A_c}\mathcal{F}(i',j),$$

- $\forall i\in A_c$,

$$H^*(i,B)=\langle a_i\rangle,\ C^*(i,B)=\bigoplus_{j\in A_c}\mathcal{F}(i,j)[-1], I^*(i,B)=\langle a_i+b_i\rangle\oplus\Big(\bigoplus_{j\in A_c}\mathcal{F}(i,j')\Big)/\langle a_i,b_i\rangle,$$

$$H^*(B,i)=\langle e_i\rangle,\ I^*(B,i)=\Big(\bigoplus_{j\in A_c}\mathcal{F}(j,i)[1]\Big)/\langle e_i\rangle,\ C^*(B,i)=\bigoplus_{j\in A_c}\mathcal{F}(j',i),$$

- *For* B,

$$H^*(B,B)=\langle e_B\rangle^0\oplus\langle b_1\rangle^1 \text{ where } e_B=\sum_{i=1}^{n}(e_i+e_{i'}),$$

$$I^*(B,B)=\langle a_i+b_i\mid 1\leq i\leq n\rangle\oplus\langle a_i+b_{i+1}\mid 1<i\leq n\rangle\oplus$$
$$\bigoplus_{i,j\in A_c}\langle(\rho a_i+\rho b_{i+1})+(a_j\rho+b_j\rho)\mid\rho\in\mathcal{F}(i',j)\rangle\oplus\Big(\bigoplus_{i,j\in A_c}\mathcal{F}(i,j')[1]\Big)/\langle a_i,b_i\rangle,$$

$$C^*(B,B)=\langle e_i\mid 1\leq i\leq n\rangle\oplus\langle e_{i'}\mid 1<i\leq n\rangle\oplus\bigoplus_{i,j\in A_c}\mathcal{F}(i',j)[-1]\oplus\Big(\bigoplus_{i,j\in A_c}\mathcal{F}(i',j')\Big)/\langle e_{i'}\rangle,$$

where a quotient of the form $V/\langle\rho_1,\dots,\rho_s\rangle$ is identified with the subspace of V generated by all the boundary paths except the ρ_l.

Figure 3.4.1 gives examples of configurations of arcs in $\hat{A}$ that induce the decomposition of the preceding lemma.

Proof: Recall the basis and notations introduced in Subsection 3.4.1.

- For $i,j\in A$, $\mathcal{A}(i,j)=\mathcal{F}(i,j)$ and the differential is zero.

- For $j\in A_s$, $\mathcal{A}(j,B)=\bigoplus_{i=1}^{n}\mathcal{F}(j,i)[-1]\oplus\bigoplus_{i=1}^{n}\mathcal{F}(j,i')$. For $\rho\in\mathcal{F}(j,i')$, $\tilde{\mu}^1(\rho)=0$, and for $\rho'\in\mathcal{F}(j,i)$, $\tilde{\mu}^1(\rho')=(-1)^{\|\rho'\|}(a_i\rho'+b_i\rho')$, where one of the two terms is zero and the other is not, depending on the endpoint of ρ'. This gives a decomposition of $\mathcal{A}(j,B)$ as the sum of the kernel of $\tilde{\mu}^1$ and a complement. Since each boundary path $\rho\in\mathcal{F}(j,i')$ factors as $\rho=a_i\rho'$ or $\rho=b_{i+1}\rho'$, by construction of the lifted dissection $\hat{A}$, every morphism of the kernel is in the image of the differential.

- Similarly, for $j\in A_s$, $\mathcal{A}(B,j)=\bigoplus_{i=1}^{n}\mathcal{F}(i,j)[1]\oplus\bigoplus_{i=1}^{n}\mathcal{F}(i',j)$ and the decomposition of a path $\rho\in\mathcal{F}(i,j)$ as $\rho=\rho' a_i$ or $\rho=\rho' b_i$ gives the description.

- For $i\in A_c$, $\mathcal{A}(i,B)=\Big(\bigoplus_{j=1}^{n}\mathcal{F}(i,j)[-1]\Big)\oplus\Big(\bigoplus_{j=1}^{n}\mathcal{F}(i,j')\Big)$ is still a decomposition between the kernel and a complement. A non trivial path ρ in $\mathcal{F}(i,j)$ is sent by $\tilde{\mu}^1$ to $(-1)^{\|\rho\|}a_j\rho$ or $(-1)^{\|\rho\|}b_j\rho$,

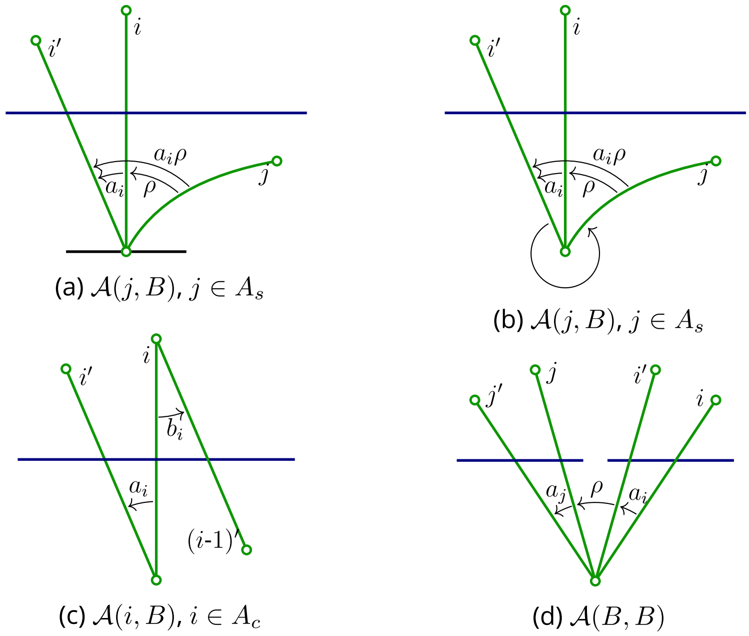


Figure 3.4.1: Examples of arc configurations in the lifted dissection $\hat{A}$.

and the trivial path e_i is sent to $-(a_i+b_i)$. Thus the image is as described and the homology is $\langle a_i, b_i \mid a_i+b_i=0\rangle$, for which we choose a_i as a basis.

- Similarly for $i\in A_c$, $\mathcal{A}(B,i) = \Big(\bigoplus_{j=1}^{n} \mathcal{F}(j,i)[1]\Big) \oplus \Big(\bigoplus_{j=1}^{n} \mathcal{F}(j',i)\Big)$ is a decomposition between the kernel and a complement. A (non trivial) path ρ in $\mathcal{F}(j',i)$ is sent by $\tilde{\mu}^1$ to $(-1)^{\|\rho\|}\rho a_j$ or $(-1)^{\|\rho\|}\rho b_{j+1}$, thus the only boundary path which is not in the image is e_i.

- We now give a decomposition for $\mathcal{A}(B,B)$. Recall the standard basis:

$$\mathcal{A}(B,B) = \bigoplus_{i,j=1}^{n} \mathcal{F}(i',j)[-1] \oplus \big(\mathcal{F}(i,j)\oplus\mathcal{F}(i',j')\big) \oplus \mathcal{F}(i,j')[1].$$

We first describe the kernel. The space $\bigoplus_{i,j=1}^{n} \mathcal{F}(i,j')[1]$ is included in the kernel. Elements of $\bigoplus_{i,j=1}^{n} \mathcal{F}(i',j)[-1]$ are sent to $\bigoplus_{i,j=1}^{n} \mathcal{F}(i,j)\oplus\mathcal{F}(i',j')$ by $\tilde{\mu}^1$, and elements of $\bigoplus_{i,j=1}^{n} \mathcal{F}(i,j)\oplus\mathcal{F}(i',j')$ are sent to $\bigoplus_{i,j=1}^{n} \mathcal{F}(i,j')[1]$. Moreover, no element of $\bigoplus_{i,j=1}^{n} \mathcal{F}(i',j)[-1]$ is sent to zero.

It remains to study elements ω of $\bigoplus_{i,j=1}^{n} \mathcal{F}(i,j)\oplus\mathcal{F}(i',j')$ which are in the kernel. Recall that

$$\forall \rho\in\mathcal{F}(i,j),\ \tilde{\mu}^1(\rho) = (-1)^{|\rho|}(a_j\rho + b_j\rho), \text{ and } \forall\rho\in\mathcal{F}(i',j'),\ \tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}(\rho a_i + \rho b_{i+1}).$$

This shows that for a decomposition $\omega = \sum_{i=1}^{n}(\lambda_i e_i + \mu_i e_{i'}) + \omega'$ where ω' is not supported on the trivial paths, both $\sum_{i=1}^{n}(\lambda_i e_i + \mu_i e_{i'})$ and ω' must be in the kernel. In ω', each term $\rho a_i \in \mathcal{F}(i,j)$ with $a_j\rho$ non zero must be compensated by a term $a_j\rho \in \mathcal{F}(i',j')$, and similarly for the other terms. Thus ω' is in $\bigoplus_{i,j=1}^{n} \langle \rho a_i + \rho b_{i+1} + a_j\rho + b_j\rho \mid \rho \in \mathcal{F}(i',j)\rangle$. One can see by applying $\tilde{\mu}^1$ on $\sum_{i=1}^{n}(\lambda_i e_i + \lambda_{i'} e_{i'})$, that one must have $\lambda_i = \lambda_{j'}$ for $i,j \in \{1,\dots,n\}$. This shows that the kernel $K^*(B,B)$ is

$$K^*(B,B) = \langle e_B\rangle \oplus \bigoplus_{i,j=1}^{n} \langle \rho a_i + \rho b_{i+1} + a_j\rho + b_j\rho \mid \rho \in \mathcal{F}(i',j)\rangle \oplus \bigoplus_{i,j=1}^{n} \mathcal{F}(i,j')[1].$$

The space $C^*(B,B)$ given in the lemma is a complement since for $\rho \in \mathcal{F}(i',j)$, one can get elements $\rho a_i + \rho b_{i+1}$ via $(\rho a_i + \rho b_{i+1} + a_j\rho + b_j\rho) - (a_j\rho + b_j\rho)$. We now describe the image. First, for $\rho \in \mathcal{F}(i',j)$, $\tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}(\rho a_i + \rho b_{i+1} + a_j\rho + b_j\rho)$. For $i,j \in \{1,\dots,n\}$, a boundary path in $\mathcal{F}(i,j')$ which is not a_i or b_i is in the image. Including the image of $\tilde{\mu}^1$ restricted to $\langle e_i, e_{i'} \mid 1 \le i \le n\rangle$, one gets for $n \ge 2$,

$$\begin{aligned} I^*(B,B) = \langle \{a_i + b_i \mid 1 \le i \le n\} \cup \{a_i + b_{i+1} \mid 1 \le i \le n\}\rangle \oplus \\ \bigoplus_{i,j \in A_c} \langle (\rho a_i + \rho b_{i+1}) + (a_j\rho + b_j\rho) \mid \rho \in \mathcal{F}(i',j)\rangle \oplus \Big(\bigoplus_{i,j\in A_c} \mathcal{F}(i,j')[1]\Big)/\langle a_i, b_i\rangle, \end{aligned}$$

The homology is then $\langle e_B\rangle \oplus \langle \{a_i, b_i \mid 1 \le i \le n\} \mid \{a_i + b_i, a_i + b_{i+1} \mid 1 \le i \le n\}\rangle$, and we choose $\{e_B, b_1\}$ as a basis. Moreover, we can choose $\{a_i + b_i \mid 1 \le i \le n\} \cup \{a_i + b_{i+1} \mid 1 < i \le n\}$ as a basis of $\langle \{a_i + b_i \mid 1 \le i \le n\} \cup \{a_i + b_{i+1} \mid 1 \le i \le n\}\rangle$ since, if $n \ge 2$, $a_1 + b_2 = \sum_{i=1}^{n}(a_i + b_i) - \sum_{i=2}^{n}(a_i + b_{i+1})$, and for an arbitrary linear combination we have

$$\sum_{i=1}^{n} \lambda_i(a_i + b_i) + \sum_{i=2}^{n} \mu_i(a_i + b_{i+1}) = 0 \iff \lambda_1 a_1 + \sum_{i=2}^{n}(\lambda_i + \mu_i)a_i + \lambda_2 b_2 + \sum_{i=2}^{n}(\lambda_{i+1} + \mu_i)b_{i+1} = 0.$$

Thus the λ_i and μ_i are zero. The case $n = 1$ is trivial.

□

For the convenience of future computations, we write down the following decomposition for the endomorphism ring of B.

Lemma 3.4.4 *The following gives a decomposition of graded vector spaces*

$$\mathcal{A}(B,B) = H^*(B,B) \oplus I^*(B,B) \oplus C^*(B,B),$$

where $I^*(B,B)$ denotes the image of $\tilde{\mu}^1$, $H^*(B,B)$ is a complement of $I^*(B,B)$ in the kernel of $\tilde{\mu}^1$, and $C^*(B,B)$ is a complement of the kernel of $\tilde{\mu}^1$:

$$\begin{aligned} &H^*(B,B) = \langle e_B\rangle^0 \oplus \langle b_1\rangle^1 \text{ where } e_B = \sum_{i=1}^{n}(e_i + e_{i'}), \\ &I^*(B,B) = \langle b_1 + a_1, \dots, b_1 + a_n, b_1 - b_2, \dots, b_1 - b_n\rangle \oplus \\ &\qquad \bigoplus_{i,j\in A_c} \langle (\rho a_i + \rho b_{i+1}) + (a_j\rho + b_j\rho) \mid \rho \in \mathcal{F}(i',j)\rangle \oplus \Big(\bigoplus_{i,j\in A_c} \mathcal{F}(i,j')[1]\Big)/\langle a_i, b_i\rangle, \end{aligned}$$

$$C^*(B,B) = \langle \sum_{i=1}^{j-1}(e_i + e_{i'}) + e_j \mid 1 \leq j \leq n\rangle \oplus$$
$$\langle \sum_{i=j}^{n}(e_i + e_{i'}) \mid 1 < j \leq n\rangle \oplus \bigoplus_{i,j\in A_c} \mathcal{F}(i',j)[-1] \oplus \Big(\bigoplus_{i,j\in A_c} \mathcal{F}(i',j')\Big)/\langle e_{i'}\rangle,$$

where a quotient of the form $V/\langle \rho_1,\dots,\rho_s\rangle$ is identified with the subspace of V generated by all boundary paths except the ρ_l.

Proof: We can choose the set $\{b_1 + a_1, ..., b_1 + a_n, b_1 - b_2, ..., b_1 - b_n\}$ as a basis of the space $\langle a_i + b_i \mid 1 \leq i \leq n\rangle \oplus \langle a_i + b_{i+1} \mid 1 < i \leq n\rangle$ since for $i \geq 2$, $a_i + b_i = (b_1 + a_i) - (b_1 - b_i)$ and for $1 < i < n$, $a_i + b_{i+1} = (b_1 - a_i) - (b_1 - b_{i+1})$. Moreover, the set $\mathcal{B} \cup \{e_B\}$, where

$$\mathcal{B} = \{\sum_{i=1}^{j-1}(e_i + e_{i'}) + e_j \mid 1 \leq j \leq n\} \cup \{\sum_{i=j}^{n}(e_i + e_{i'}) \mid 1 < j \leq n\},$$

generates $\langle e_i, e_{i'} \mid 1 \leq i \leq n\rangle$. First for $1 \leq j \leq n$,

$$e_j = \big(\sum_{i=1}^{j-1}(e_i + e_{i'}) + e_j\big) + \sum_{i=j}^{n}(e_i + e_{i'}) - e_B.$$

Then for $1 \leq j < n$,

$$e_{j'} = \big(\sum_{i=1}^{j}(e_i + e_{i'}) + e_{j+1}\big) - e_{j+1} - \big(\sum_{i=1}^{j-1}(e_i + e_{i'}) + e_j\big),$$

and $e_{n'} = e_B - \big(\sum_{i=1}^{n-1}(e_i + e_{i'}) + e_n\big)$. It is a basis since it has the desired cardinality.

□

Example 3.4.5

(1) *For $\hat{S}$ and $\hat{A}$ as in Figure 3.3.6, $A = \{1,2,3\} = A_c$ where c is the only conical singularity of S. Let $i \in \{1,2,3\}$. The decomposition given by the preceding lemma is:*

$$H^*(i,B) = \langle a_i\rangle,\ C^*(i,B) = \langle e_i\rangle,\ I^*(i,B) = \langle a_i + b_i\rangle,$$
$$H^*(B,i) = \langle e_i\rangle,\ C^*(B,i) = 0,\ I^*(B,i) = 0,$$
$$H^*(B,B) = \langle e_B, b_1\rangle, I^*(B,B) = \langle b_1 + a_1, b_1 + a_2, b_1 + a_3, b_1 - b_2, b_1 - b_3\rangle,$$
$$C^*(B,B) = \langle e_1, e_1 + e_{1'} + e_2, e_1 + e_{1'} + e_2 + e_{2'} + e_3\rangle \oplus \langle e_2 + e_{2'} + e_3 + e_{3'}, e_3 + e_{3'}\rangle.$$

(2) *For $\hat{S}$ and $\hat{A}$ as in Figure 3.3.7, $A = A_s \sqcup A_c$ with $A_s = \{0\}$ and $A_c = \{1,2\}$. For x, y two consecutive arrows of the cycle $(c_2' b_2 c_1' a_1 c_0)$, we write $(x \dots y)$ to designate the path that goes around it once, starting at y. The decomposition is:*

$$I^*(0,B) = \langle (a_1 \dots c_1')^k a_1 c_0, (b_2 \dots c_2')^k b_2 c_1' a_1 c_0 \mid k \in \mathbb{N}\rangle,$$

$$C^*(0,B) = \langle (c_0 \dots a_1)^k c_0, (c_1' \dots b_2)^k c_1' a_1 c_0 \mid k \in \mathbb{N} \rangle,$$
$$I^*(B,0) = \langle (c_2' \dots c_0)^k c_2' b_2 c_1' a_1, (c_2' \dots c_0)^k c_2' b_2 \mid k \in \mathbb{N} \rangle,$$
$$C^*(B,0) = \langle (c_2' \dots c_0)^k c_2', (c_2' \dots c_0)^k c_2' b_2 c_1' \mid k \in \mathbb{N} \rangle,$$

$$H^*(1,B) = \langle a_1 \rangle, C^*(1,B) = \langle (c_0 \dots a_1)^k, (c_1' \dots b_2)^k c_1' a_1 \mid k \in \mathbb{N} \rangle,$$
$$I^*(1,B) = \langle a_1 + b_1, (a_1 \dots c_1')^{k+1} a_1, (b_2 \dots c_2')^k b_2 c_1' a_1 \mid k \in \mathbb{N} \rangle,$$
$$H^*(B,1) = \langle e_1 \rangle, C^*(B,1) = \langle (c_0 \dots a_1)^k c_0 c_2', (c_0 \dots a_1)^k c_0 c_2' b_2 c_1' \mid k \in \mathbb{N} \rangle,$$
$$I^*(B,1) = \langle (c_0 \dots a_1)^{k+1}, (c_0 \dots a_1)^k c_0 c_2' b_2 \mid k \in \mathbb{N} \rangle,$$

$$H^*(2,B) = \langle a_2 \rangle, C^*(2,B) = \langle (c_0 \dots a_1)^k c_0 c_2' b_2, (c_1' \dots b_2)^k \mid k \in \mathbb{N} \rangle,$$
$$I^*(2,B) = \langle a_2 + b_2, (b_2 \dots c_2')^{k+1} b_2, (a_1 \dots c_1')^k a_1 c_0 c_2' b_2 \mid k \in \mathbb{N} \rangle,$$
$$H^*(B,2) = \langle e_2 \rangle, C^*(B,2) = \langle (c_1' \dots b_2)^k c_1', (c_1' \dots b_2)^k c_1' a_1 c_0 c_2' \mid k \in \mathbb{N} \rangle,$$
$$I^*(B,2) = \langle (c_1' \dots b_2)^k c_1' a_1, (c_1' \dots b_2)^{k+1} \mid k \in \mathbb{N} \rangle,$$

$$H^*(B,B) = \langle e_B, b_1 \rangle, I^*(B,B) = \langle b_1 + a_1, b_1 + a_2, b_1 - b_2 \rangle \oplus$$
$$\langle (c_0 \dots a_1)^k c_0 c_2' (a_1 + b_2) + (a_1 + b_1)(c_0 \dots a_1)^k c_0 c_2',$$
$$(c_0 \dots a_1)^k c_0 c_2' b_2 c_1' (a_1 + b_2) + (a_1 + b_1)(c_0 \dots a_1)^k c_0 c_2' b_2 c_1',$$
$$(c_1' \dots b_2)^k c_1' (a_1 + b_2) + (a_2 + b_2)(c_1' \dots b_2)^k c_1',$$
$$(c_1' \dots b_2)^k c_1' a_1 c_0 c_2' (a_1 + b_2) + (a_2 + b_2)(c_1' \dots b_2)^k c_1' a_1 c_0 c_2' \mid k \in \mathbb{N} \rangle \oplus$$
$$\langle (a_1 \dots c_1')^{k+1} a_1, (b_2 \dots c_2')^k b_2 c_1' a_1, (b_2 \dots c_2')^{k+1} b_2, (a_1 \dots c_1')^k a_1 c_0 c_2' b_2 \mid k \in \mathbb{N} \rangle,$$
$$C^*(B,B) = \langle e_1, e_1 + e_{1'} + e_2 \rangle \oplus \langle e_2 + e_{2'} \rangle \oplus$$
$$\langle (c_0 \dots a_1)^k c_0 c_2', (c_0 \dots a_1)^k c_0 c_2' b_2 c_1', (c_1' \dots b_2)^k c_1', (c_1' \dots b_2)^k c_1' a_1 c_0 c_2' \mid k \in \mathbb{N} \rangle \oplus$$
$$\langle (a_1 \dots c_1')^{k+1}, (b_2 \dots c_2')^{k+1}, (b_2 \dots c_2')^k b_2 c_1', (a_1 \dots c_1')^k a_1 c_0 c_2' \mid k \in \mathbb{N} \rangle.$$

Setting 3.4.6 *Let (x,y) be a pair of objects in $\mathcal{A}$. We set $f^1 : H^*\mathcal{A}(x,y) \to \mathcal{A}(x,y)$ to be the inclusion and $g^1 : \mathcal{A}(x,y) \to H^*\mathcal{A}(x,y)$ to be the projection with respect to the basis given in Lemma 3.4.3 and Lemma 3.4.4 for $\mathcal{A}(B,B)$. We define an endomorphism T^1 (of graded vector space) of $\mathcal{A}(x,y)$ of degree -1, which is zero on $H^*(x,y) \oplus C^*(x,y)$.*

- *For $(x,y) = (j,B)$ with $j \in A_s$: $T^1(a_i\rho) = (-1)^{|\rho|}\rho$ and $T^1(b_i\rho) = (-1)^{|\rho|}\rho$,*

- *For $(x,y) = (B,j)$ with $j \in A_s$: $T^1(\rho a_i) = (-1)^{|\rho|}\rho$ and $T^1(\rho b_i) = (-1)^{|\rho|}\rho$,*

- *For $(x,y) = (i,B)$ with $i \in A_c$:*
  $T^1(a_i + b_i) = e_i$ and $T^1(a_k\rho) = T^1(b_k\rho) = (-1)^{|\rho|}\rho$ for $k \in A_c$ and ρ non trivial in $\mathcal{F}(i,k)$,

- *For $(x,y) = (B,i)$ with $i \in A_c$: $T^1(\rho a_k) = T^1(\rho b_k) = (-1)^{|\rho|}\rho$ for $k \in A_c$,*

- *For the case $(x,y) = (B,B)$:*

  - *For $1 \le j \le n$, $T^1(b_1 + a_j) = -(\sum_{i=1}^{j-1}(e_i + e_{i'}) + e_j)$,*

- *For* $1 < j \leq n$, $T^1(b_1 - b_j) = \sum_{i=j}^{n}(e_i + e_{i'})$,
- *For* $\rho \in \mathcal{F}(i', j)$, $T^1((\rho a_i + \rho b_{i+1}) + (a_j \rho + b_j \rho)) = (-1)^{|\rho|}\rho$,
- *For a path* ρ *in* $\mathcal{F}(i, i')/\langle a_i \rangle$, $\mathcal{F}(i, (i-1)')/\langle b_i \rangle$ *or* $\mathcal{F}(i, j')$, *(or in* $\mathcal{F}(1, 1')/\langle a_1, b_1 \rangle$ *when* $n = 1$*),* ρ *is of the form* $\rho = \rho' a_i$ *or* $\rho = \rho' b_i$, *with* ρ' *non trivial. We set*

$$T^1(\rho' a_i) = T^1(\rho' b_i) = (-1)^{|\rho'|}\rho'.$$

Using the formula for $\tilde{\mu}^1$ *given in subsubsection 3.4.1.2, one can check that* T^1 *satisfies Equation 3.5. It is immediately true for elements of* $H^*(x, y)$ *since* T^1 *send them to zero. For elements of* $I^*(x, y)$ *the condition becomes* $\tilde{\mu}^1 T^1 = -\mathrm{id}$, *and for those of* $C^*(x, y)$ *it becomes* $T^1 \tilde{\mu}^1 = -\mathrm{id}$ *since* T^1 *also sends them to zero.*

For example in the case $(x, y) = (j, B)$ *one has:*

$$\tilde{\mu}^1 T^1(a_i \rho) = (-1)^{|\rho|}\tilde{\mu}^1(\rho) = (-1)^{|\rho|}(-1)^{\|\rho\|}a_i \rho,$$
$$\text{and } T^1 \tilde{\mu}^1(\rho) = (-1)^{\|\rho\|}T^1(a_i \rho + b_i \rho) = (-1)^{\|\rho\|}(-1)^{|\rho|}\rho.$$

The other cases are similar except for $(x, y) = (B, B)$ *which comes from the following formulas:*

- *For* $1 \leq j \leq n$, $\tilde{\mu}^1(\sum_{i=1}^{j-1}(e_i + e_{i'}) + e_j) = \sum_{i=1}^{j-1}((a_i + b_i) - (a_i + b_{i+1})) + (a_j + b_j) = b_1 + a_j$,
- *For* $1 < j \leq n$, $\tilde{\mu}^1(\sum_{i=j}^{n}(e_i + e_{i'})) = \sum_{i=j}^{n}((a_i + b_i) - (a_i + b_{i+1})) = -(b_1 - b_j)$,
- *For* $\rho \in \mathcal{F}(i', j)$,

$$\begin{aligned}\tilde{\mu}^1(\rho) &= \tilde{\mu}^2(s \otimes \rho, s^{-1} \otimes (a_i + b_i)) + \tilde{\mu}^2(s^{-1} \otimes (a_j + b_j), s \otimes \rho) \\ &= (-1)^{\|\rho\|}s^0 \otimes \hat{\mu}^2(\rho, a_i + b_i) - s^0 \otimes \hat{\mu}^2(a_j + b_j, \rho) \\ &= (-1)^{\|\rho\|}(\rho a_i + \rho b_i) + (-1)^{\|\rho\|}(a_j \rho + b_j \rho),\end{aligned}$$

- $\tilde{\mu}^1(\rho') = \tilde{\mu}^2(s^0 \otimes \rho', s^{-1} \otimes (a_i + b_i)) = (-1)^{\|\rho\|}s^{-1} \otimes \hat{\mu}^2(\rho', a_i + b_i) = (-1)^{\|\rho'\|}(\rho' a_i + \rho' b_i)$.

This will allow us to apply Proposition 3.4.2 to this setting.

Notations 3.4.7 *We first rename the basis of* $H^*\mathcal{A}$ *before describing the minimal model. For* $i \in A_c$:

- *Let* $t_i := a_i \in H^0(i, B)$,
- *Let* $z_i := e_i \in H^1(B, i)$,
- *let* $x := b_1 \in H^1(B, B)$.

Proposition 3.4.8 *Let* $\mathcal{A}$ *be the* A_∞*-category defined in Setting 3.4.1. The following higher multiplications* μ^n *on* $H^*\mathcal{A}$ *give a minimal model for* $\mathcal{A}$.

Let $j \in A$ *and* $i, k \in A_c$. *We give a complete list of the non-vanishing* $\mu^n(x_n, \ldots, x_1)$ *for* $x_1, \ldots, x_n$ *passing through* $(X_0, X_1, \ldots, X_n)$.

$(n = 2)$

- *On the arc collection A, μ^2 restricts to $\hat{\mu}^2$,*
- (B,i,B)*: For $z_i \in \mathcal{A}(B,i)$, $t_i \in \mathcal{A}(i,B)$, $\mu^2(t_i,z_i) = x$.*

$(n=3)$

- (i,B,i,j)*: For ρa_i in $\mathcal{F}(i,j)$, $\mu^3(\rho a_i, z_i, t_i) = -\rho a_i$,*
- (j,i,B,i)*: For ρ a non trivial path in $\mathcal{F}(j,i)$ such that $a_i\rho$ is non zero, $\mu^3(z_i,t_i,\rho) = (-1)^{\|\rho\|}\rho$,*
- (k,B,i,B)*: For $1 \leq k \leq i-1$, $\mu^3(t_i,z_i,t_k) = t_k$,*
- (B,i,B,k)*: For $1 \leq k \leq i$, $\mu^3(z_k,t_i,z_i) = z_k$,*
- (B,i,B,B)*: $\mu^3(x,t_i,z_i) = x$.*

The elements $e_1,\dots,e_n,e_B$ are units.

The next subsection is devoted to the proof of Proposition 3.4.8, and the main part of the text resume in Section 3.5.

3.4.4 The homotopy transfer

We prove Proposition 3.4.8. For this we place ourselves as in Lemma 3.4.3 and Lemma 3.4.4 for the decomposition of $\mathcal{A}(B,B)$, and as in Setting 3.4.6. We will apply recursively Equations 3.6 of Proposition 3.4.2.

For each iteration we give a list of the non vanishing terms:

$$\Sigma(n) := \sum_r \sum_{s_1,\dots,s_r} \tilde{\mu}^r(f^{s_r}(x_n,\dots,x_{n-s_r+1}),\dots,f^{s_1}(x_{s_1},\dots,x_1)),$$

for all possible sequences $x_1,\dots,x_n$ of composable morphisms of $H^*\mathcal{A}$ passing through the sequence of objects $(X_0,X_1,\dots,X_n)$. Notes that since $\hat{\mu}^n = 0$ for $n \geq 3$, $\tilde{\mu}^n$ also vanishes for $n \geq 3$ and $\Sigma(n)$ becomes:

$$\Sigma(n) = \sum_{1\leq s\leq n-1} \tilde{\mu}^2(f^{n-s}(x_n,\dots,x_{s+1}),f^s(x_s,\dots,x_1)).$$

We then apply g^1 (resp. T^1) to compute μ^n (resp. f^n). Recall that the rules for computing $\tilde{\mu}^n$ are given by Equations 3.1 and 3.3. When taking a morphism ρ in $\mathcal{F}$ we always mean a basis element, that is, a non-zero path.

• $n=2, \Sigma(2) = \tilde{\mu}^2(f^1(x_2),f^1(x_1))$.
Let $j,l,m \in A$ and $i,k \in A_c$.

- $(X_0,X_1,X_2) = (j,l,m)$:
  - ▷ For $\rho \in \mathcal{F}(j,l)$ and $\rho' \in \mathcal{F}(l,m)$, $\tilde{\mu}^2(\rho',\rho) = \hat{\mu}^2(\rho',\rho) = (-1)^{|\rho|}\rho'\rho$.
    Thus $\mu^2(\rho',\rho) = (-1)^{|\rho|}g^1(\rho'\rho) = (-1)^{|\rho|}\rho'\rho$ and $f^2(\rho',\rho) = (-1)^{|\rho|}T^1(\rho'\rho) = 0$.

- (j,i,B):
  - ▷ For $\rho \in \mathcal{F}(j,i)$, $\tilde{\mu}^2(a_i,\rho) = \tilde{\mu}^2(s^0 \otimes a_i, s^0 \otimes \rho) = s^0 \otimes \hat{\mu}^2(a_i,\rho) = (-1)^{|\rho|}a_i\rho$.
    $\rightarrow$ If $j = i$ and $\rho = e_i$, then by applying g^1, one get $\mu^2(a_i,e_i) = a_i$, and by applying T^1, $f^2(a_i,e_i) = 0$ since a_i is in the homology.
    Otherwise, if ρ is non trivial and $a_i\rho$ is non zero, $\mu^2(a_i,\rho) = 0$ and
    $$f^2(a_i,\rho) = (-1)^{|\rho|}T^1(a_i\rho) = (-1)^{|\rho|}(-1)^{|\rho|}\rho = \rho.$$
- (B,i,j):
  - ▷ For $\rho \in \mathcal{F}(i,j)$, $\tilde{\mu}^2(\rho,e_i) = \tilde{\mu}^2(s^0 \otimes \rho, s^{-1} \otimes e_i) = (-1)^{\|\rho\|}s^{-1} \otimes \rho$.
    $\rightarrow$ If $i = j$ and $\rho = e_i$, $\mu^2(e_i,e_i) = -e_i$ and $f^2(e_i,e_i) = 0$.
    Otherwise $\rho = \rho' a_i + \rho' b_i$ for some ρ', and $\mu^2(\rho,e_i) = 0$,
    $$f^2(\rho,e_i) = (-1)^{\|\rho\|}T^1(\rho' a_i + \rho' b_i) = (-1)^{\|\rho\|}(-1)^{|\rho'|}\rho' = (-1)^{(|\rho|-1)+|\rho|}\rho' = -\rho'.$$
- (i,B,k):
  - ▷ $\tilde{\mu}^2(e_k,a_i) = \tilde{\mu}^2(s^{-1} \otimes e_k, s^0 \otimes a_i) = 0$.
- (i,B,B):
  - ▷ $\tilde{\mu}^2(e_B,a_i) = a_i$.
    $\rightarrow \mu^2(e_B,a_i) = a_i$, $f^2(e_B,a_i) = 0$.
  - ▷ $\tilde{\mu}^2(b_1,a_i) = \tilde{\mu}^2(s^{-1} \otimes b_1, s^0 \otimes a_i) = 0$.
- (B,B,i):
  - ▷ $\tilde{\mu}^2(e_i,e_B) = e_i$.
    $\rightarrow \mu^2(e_i,e_B) = e_i$, $f^2(e_i,e_B) = 0$.
  - ▷ $\tilde{\mu}^2(e_i,b_1) = 0$.
- (B,i,B):
  - ▷ $\tilde{\mu}^2(a_i,e_i) = \tilde{\mu}^2(s^0 \otimes a_i, s^{-1} \otimes e_i) = -s^{-1} \otimes a_i = b_1 - (b_1 + a_i)$.
    $\rightarrow$ Applying g^1 gives $\mu^2(a_i,e_i) = b_1$, and applying T^1 gives
    $$f^2(a_i,e_i) = -T^1(b_1 + a_i) = \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i.$$
  - ▷ $\tilde{\mu}^2(a_i,e_i) = \tilde{\mu}^2(s^0 \otimes a_i, s^{-1} \otimes e_i) = -s^{-1} \otimes a_i = b_1 - (b_1 + a_i)$.
    $\rightarrow$ Applying g^1 gives $\mu^2(a_i,e_i) = b_1$, and applying T^1 gives
    $$f^2(a_i,e_i) = -T^1(b_1 + a_i) = \sum_{j=1}^{i-1}(e_j + e_{j'}) + e_i \text{ if } i \neq 1,$$
    and $f^2(a_1,e_1) = -T^1(b_1 + a_1) = e_1 - e_B$.

- (B, B, B):

  ▷ $\tilde{\mu}^2(e_B, e_B) = e_B$, $\tilde{\mu}^2(b_1, e_B) = b_1$, $\tilde{\mu}^2(e_B, b_1) = -b_1$, $\tilde{\mu}^2(b_1, b_1) = 0$.

Let's summarize the non vanishing cases. Let $j \in A$ and $i \in A_c$.

- On the arc collection A, μ^2 restricts to $\hat{\mu}^2$,
- The element e_B and the e_j's in $H^*\mathcal{A}(j, j)$ are units: $\mu^2(x, e) = x$ and $\mu^2(e, x) = (-1)^{|x|_\mathcal{A}} x$, for e one of these elements and x in $H^*\mathcal{A}$ respectively post and pre-composable,
- (B, i, B): For $e_i \in \mathcal{A}(B, i)$, $a_i \in \mathcal{A}(i, B)$, $\mu^2(a_i, e_i) = b_1$.

The non vanishing f^2 are:

- (j, i, B): For $\rho \in \mathcal{F}(j, i)$ non trivial such that $a_i\rho$ is non zero, $f^2(a_i, \rho) = \rho$,
- (B, i, j): For $\rho \in \mathcal{F}(i, j)$ decomposing as $\rho = \rho' a_i + \rho' b_i$ for some ρ', $f^2(\rho, e_i) = -\rho'$,
- (B, i, B): $f^2(a_i, e_i) = \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i$.

• $n = 3$, $\Sigma(3) = \tilde{\mu}^2(f^2(x_3, x_2), f^1(x_1)) + \tilde{\mu}^2(f^1(x_3), f^2(x_2, x_1))$. Let $j, l \in A$ and $i, k \in A_c$.
· We first consider all the cases where $f^2(x_3, x_2)$ is non zero.
Let $\rho \in \mathcal{F}(j, i)$ non trivial such that $a_i\rho$ is non zero. Recall that $f^2(a_i, \rho) = \rho$.

- (l, j, i, B):

  ▷ For $\gamma \in \mathcal{F}(l, j)$, $\tilde{\mu}^2(f^2(a_i, \rho), \gamma) = \tilde{\mu}^2(s^1 \otimes \rho, s^0 \otimes \gamma) = (-1)^{|\gamma|} s^1 \otimes \rho\gamma$. In this case, since $f^2(\rho, \gamma) = 0$, $\Sigma(3) = \tilde{\mu}^2(f^2(a_k, \rho), \gamma)$. Both g^1 and T^1 vanish on this element.

- (B, i, k, B):
  Let $\rho \in \mathcal{F}(i, k)$ decomposing as $\rho = \rho' a_i + \rho' b_i$ for some ρ', and such that $a_k\rho$ is non zero.

  ▷ First, $\tilde{\mu}^2(f^2(a_k, \rho), e_i) = \tilde{\mu}^2(s^1 \otimes \rho, s^{-1} \otimes e_i) = (-1)^{\|\rho\|} s^0 \otimes \rho$.
  Then, $\tilde{\mu}^2(a_k, f^2(\rho, e_i)) = -\tilde{\mu}^2(s^0 \otimes a_k, s^0 \otimes \rho') = -(-1)^{|\rho'|} s^0 \otimes a_k\rho'$, and

  $$\Sigma(3) = \tilde{\mu}^2(f^2(a_k, \rho), e_i) + \tilde{\mu}^2(a_k, f^2(\rho, e_i)) = (-1)^{\|\rho\|}(\rho' a_i + \rho' b_i + a_k\rho')$$

  $\to g^1$ vanishes on this element, and applying T^1 gives

  $$f^3(a_k, \rho, e_i) = (-1)^{\|\rho\|}(-1)^{|\rho'|}\rho' = -\rho'.$$

Let $\rho \in \mathcal{F}(i, j)$ decomposing as $\rho = \rho' a_i + \rho' b_i$ for some ρ', and let $e_i \in \mathcal{A}(B, i)$. Recall that $f^2(\rho, e_i) = -\rho'$.
Since $f^2(e_i, x_1) = 0$ for all x_1, $\Sigma(3) = \tilde{\mu}^2(f^2(\rho, e_i), f^1(x_1))$.

- (k, B, i, j):
  - ▷ $\tilde{\mu}^2(f^2(\rho, e_i), a_k) = -\tilde{\mu}^2(s^0 \otimes \rho', s^0 \otimes a_k) = -s^0 \otimes \rho' a_k$, which is non zero only when $\rho = \rho' a_k$.
    $\rightarrow T^1$ vanishes on this element, and applying g^1 gives $\mu^3(\rho' a_k, e_k, a_k) = -\rho' a_k$.
- (B, B, i, j):
  - ▷ $\tilde{\mu}^2(f^2(\rho, e_i), e_B) = -\tilde{\mu}^2(s^0 \otimes \rho', e_B) = -s^0 \otimes \rho'$. Both g^1 and T^1 vanish on this element.
  - ▷ $\tilde{\mu}^2(f^2(\rho, e_i), b_1) = -\tilde{\mu}^2(s^0 \otimes \rho', s^{-1} \otimes b_1) = -(-1)^{\|\rho'\|} s^{-1} \otimes \rho' b_1 = (-1)^{|\rho|} s^{-1} \otimes \rho' b_1$, which is non zero only when $\rho = \rho' b_1$. Note that ρ' is non trivial since $j \in A$.
    $\rightarrow g^1$ vanishes on this element, and applying T^1 gives
    $$f^3(\rho' b_1, e_1, b_1) = (-1)^{|\rho'|} T^1(\rho' b_1) = \rho'.$$

Let $e_i \in \mathcal{A}(B, i)$, $a_i \in \mathcal{A}(i, B)$. Recall that $f^2(a_i, e_i) = \sum_{v=1}^{i-1} (e_v + e_{v'}) + e_i$.

On the tuple $(x_1, x_2, x_3) = (x_1, e_i, a_i)$, the term $f^2(e_i, x_1)$ is necessarily zero, and we only need to compute $\Sigma(3) = \tilde{\mu}^2(f^2(a_i, e_i), f^1(x_1))$.

- (k, B, i, B):
  - ▷ $\tilde{\mu}^2(f^2(a_i, e_i), a_k) = \tilde{\mu}^2(s^0 \otimes \sum_{v=1}^{i-1} (e_v + e_{v'}) + e_i, s^0 \otimes a_k)$ which is equal to a_k if $1 \leq k \leq i-1$ and zero otherwise.
    $\rightarrow T^1$ vanishes on this element, and for $1 \leq k \leq i-1$, applying g^1 gives
    $$\mu^3(a_i, e_i, a_k) = a_k.$$
- (B, B, i, B):
  - ▷ $\tilde{\mu}^2(f^2(a_i, e_i), e_B) = \tilde{\mu}^2(\sum_{v=1}^{i-1} (e_v + e_{v'}) + e_i, e_B) = \sum_{v=1}^{i-1} (e_v + e_{v'}) + e_i$. Both g^1 and T^1 vanish on this element.
  - ▷ $\tilde{\mu}^2(f^2(a_i, e_i), b_1) = \tilde{\mu}^2(s^0 \otimes \sum_{v=1}^{i-1} (e_v + e_{v'}) + e_i, s^{-1} \otimes b_1) = 0$.

• We now consider the cases where $f^2(x_2, x_1)$ is non zero.

Let ρ be a non trivial path in $\mathcal{F}(j, i)$, decomposing as $\rho = \rho' a_j$ or $\rho = \rho' b_j$, and such that $a_i \rho$ is non zero for $a_i \in \mathcal{A}(i, B)$. Recall that $f^2(a_i, \rho) = \rho$. Since $f^2(x_3, a_i) = 0$ for all x_3, the term $\Sigma(3)$ is $\tilde{\mu}^2(f^1(x_3), f^2(a_i, \rho))$.

- (j, i, B, k) :
  - ▷ $\tilde{\mu}^2(e_k, f^2(a_i, \rho)) = \tilde{\mu}^2(s^{-1} \otimes e_k, s^1 \otimes \rho)$. This term is $-(-1)^{|\rho|} s^0 \otimes \rho = (-1)^{\|\rho\|} \rho$ if $k = i$ and zero otherwise.
    $\rightarrow T^1$ vanishes on this element and applying g^1 gives $\mu^3(e_i, a_i, \rho) = (-1)^{\|\rho\|} \rho$.

- (j, i, B, B) :
  - ▷ $\tilde{\mu}^2(e_B, f^2(a_i, \rho)) = \tilde{\mu}^2(s^0 \otimes e_B, s^1 \otimes \rho) = (-1)^{\|\rho\|} s^1 \otimes \rho$. Both g^1 and T^1 vanish on this element.
  - ▷ $\tilde{\mu}^2(b_1, f^2(a_i, \rho)) = \tilde{\mu}^2(s^{-1} \otimes b_1, s^1 \otimes \rho) = -(-1)^{|\rho|} s^0 \otimes b_1\rho = 0$.

Let $\rho \in \mathcal{F}(i, j)$ decomposing as $\rho = \rho' a_i + \rho' b_i$, and let $e_i \in H^*\mathcal{A}(B, i)$. Recall that $f^2(\rho, e_i) = -\rho'$.

- (B, i, j, l) :
  - ▷ For $\gamma \in \mathcal{F}(j, l)$, $\tilde{\mu}^2(\gamma, f^2(\rho, e_i)) = -\tilde{\mu}^2(s^0 \otimes \gamma, s^0 \otimes \rho') = (-1)^{\|\rho'\|} s^0 \otimes \gamma\rho'$. In this case, since $f^2(\gamma, \rho) = 0$, $\Sigma(3) = \tilde{\mu}^2(\gamma, f^2(\rho, e_i))$. Both g^1 and T^1 vanish on this element.
- (B, i, j, B) :
  - ▷ The case $(x_1, x_2, x_3) = (e_i, \rho, a_j)$ as already been treated before.

Let $e_i \in \mathcal{A}(B, i)$, $a_i \in \mathcal{A}(i, B)$. Recall that $f^2(a_i, e_i) = \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i$. Since $f^2(x_3, a_i) = 0$ for all x_3, $\Sigma(3) = \tilde{\mu}^2(f^1(x_3), f^2(a_i, e_i))$.

- (B, i, B, k):
  - ▷ $\tilde{\mu}^2(e_k, f^2(a_i, e_i)) = \tilde{\mu}^2(s^{-1} \otimes e_k, s^0 \otimes \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i)$ which is equal to e_k if $k \in \{1, \dots, i\}$ and zero otherwise.
    $\to T^1$ vanishes on this element, and for $1 \leq k \leq i$, applying g^1 gives $\mu^3(e_k, a_i, e_i) = e_k$.
- (B, i, B, B):
  - ▷ $\tilde{\mu}^2(e_B, f^2(a_i, e_i)) = \tilde{\mu}^2(s^0 \otimes e_B, s^0 \otimes \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i) = \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i$. Both g^1 and T^1 vanish on this element.
  - ▷ $\tilde{\mu}^2(b_1, f^2(a_i, e_i)) = \tilde{\mu}^2(s^{-1} \otimes b_1, s^0 \otimes \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i) = b_1$.
    $\to T^1$ vanishes on this element and applying g^1 gives $\mu^3(b_1, a_i, e_i) = b_1$.

We summarize here the non vanishing cases. Let $j \in A$ and $i, k \in A_c$.

- (i, B, i, j): For $\rho' a_i$ in $\mathcal{F}(i, j)$, $\mu^3(\rho' a_i, e_i, a_i) = -\rho' a_i$,
- (j, i, B, i): For ρ a non trivial path in $\mathcal{F}(j, i)$ such that $a_i\rho$ is non zero, $\mu^3(e_i, a_i, \rho) = (-1)^{\|\rho\|}\rho$,
- (k, B, i, B): For $1 \leq k \leq i-1$, $\mu^3(a_i, e_i, a_k) = a_k$,
- (B, i, B, k): For $1 \leq k \leq i$, $\mu^3(e_k, a_i, e_i) = e_k$,
- (B, i, B, B): $\mu^3(b_1, a_i, e_i) = b_1$.

The non vanishing f^3 are:

- (B,i,k,B): Let ρ in $\mathcal{F}(i,k)$ decomposing as $\rho = \rho' a_i + \rho' b_i$, and such that $a_k\rho$ is non zero for $a_k \in \mathcal{A}(k,B)$. Then $f^3(a_k,\rho,e_i) = -\rho'$,

- $(B,B,1,j)$: For $\rho' b_1 \in \mathcal{F}(1,j)$, $f^3(\rho' b_1, e_1, b_1) = \rho'$.

• $n=4$, $\Sigma(4) = \tilde{\mu}^2(f^3(x_4,x_3,x_2), f^1(x_1)) + \tilde{\mu}^2(f^2(x_4,x_3), f^2(x_2,x_1)) + \tilde{\mu}^2(f^1(x_4), f^3(x_3,x_2,x_1))$.

We will go through all the tuples (x_4,x_3,x_2,x_1) such that $(f^3(x_4,x_3,x_2), f^1(x_1))$ is non zero, then $(f^2(x_4,x_3), f^2(x_2,x_1))$ and finally $(f^1(x_4), f^3(x_3,x_2,x_1))$, without repeating the cases that already appeared. Let $j,l \in A$ and $i,k,h \in A_c$.

· Here are the cases for $(f^3(x_4,x_3,x_2), f^1(x_1))$ non zero.

For (x_2,x_3,x_4) going through (B,i,k,B). Let ρ in $\mathcal{F}(i,k)$ decomposing as $\rho = \rho' a_i + \rho' b_i$, and such that $a_k\rho$ is non zero for $a_k \in \mathcal{A}(k,B)$. Recall that $f^3(a_k,\rho,e_i) = -\rho'$.

On the tuple $(x_1,x_2,x_3,x_4) = (x_1,e_i,\rho,a_k)$, $f^2(e_i,x_1)$ is necessarily zero. The term $f^3(\rho,e_i,x_1)$ is non zero only when $(x_1,x_2,x_3,x_4) = (b_1,e_1,\rho' b_1,a_k)$ with $b_1 \in \mathcal{A}(B,B)$.

- (h,B,i,k,B):

  ▷ $\tilde{\mu}^2(f^3(a_k,\rho,e_i),a_h) = -\tilde{\mu}^2(s^1 \otimes \rho', s^0 \otimes a_h) = -s^1 \otimes \rho' a_h$. Both g^1 and T^1 vanish on this element.

- (B,B,i,k,B):

  ▷ $\tilde{\mu}^2(f^3(a_k,\rho,e_i),e_B) = -\tilde{\mu}^2(\rho',e_B)$. Both g^1 and T^1 vanish on this element.

  ▷ $\tilde{\mu}^2(f^3(a_k,\rho,e_i),b_1) = -\tilde{\mu}^2(s^1 \otimes \rho', s^{-1} \otimes b_1) = -(-1)^{\|\rho'\|} s^0 \otimes \rho' b_1$. It is non zero only when $i=1$ and $\rho = \rho' b_1$. In this case $\tilde{\mu}^2(f^3(a_k,\rho' b_1,e_1),b_1) = (-1)^{|\rho'|} s^0 \otimes \rho' b_1$.
  Moreover, $\tilde{\mu}^2(a_k, f^3(\rho' b_1,e_1,b_1)) = \tilde{\mu}^2(s^0 \otimes a_k, s^0 \otimes \rho') = (-1)^{|\rho'|} s^0 \otimes a_k\rho'$.
  Thus $\Sigma(4) = \tilde{\mu}^2(f^3(a_k,\rho' b_1,e_1),b_1) + \tilde{\mu}^2(a_k, f^3(\rho' b_1,e_1,b_1)) = (-1)^{|\rho'|}(\rho' b_1 + a_k\rho')$.
  $\to g^1$ vanishes on this element, and applying T^1 gives

$$f^4(a_k,\rho' b_1,e_1,b_1) = (-1)^{|\rho'|} T^1(\rho' b_1 + a_k\rho') = (-1)^{|\rho'|}(-1)^{|\rho'|}\rho' = \rho'.$$

For (x_2,x_3,x_4) going through $(B,B,1,j)$. Let $\rho' b_1$ in $\mathcal{F}(1,j)$. Recall that $f^3(\rho' b_1,e_1,b_1) = \rho'$.
On the tuple $(x_1,x_2,x_3,x_4) = (x_1,b_1,e_1,\rho' b_1)$, $f^2(b_1,x_1)$ and $f^3(e_1,b_1,x_1)$ are necessarily zero.

- $(i,B,B,1,j)$:

  ▷ $\tilde{\mu}^2(f^3(\rho' b_1,e_1,b_1),a_i) = \tilde{\mu}^2(s^0 \otimes \rho', s^0 \otimes a_i) = s^0 \otimes \rho' a_i = 0$.

- $(B,B,B,1,j)$:

  ▷ $\tilde{\mu}^2(f^3(\rho' b_1,e_1,b_1),e_B) = \tilde{\mu}^2(\rho',e_B)$. Both g^1 and T^1 vanish on this element.

▷ $\tilde{\mu}^2(f^3(\rho' b_1, e_1, b_1), b_1) = \tilde{\mu}^2(s^0 \otimes \rho', s^{-1} \otimes b_1) = (-1)^{\|\rho'\|} s^{-1} \otimes \rho' b_1$.
$\rightarrow g^1$ vanishes on this element and applying T^1 gives

$$f^4(\rho' b_1, e_1, b_1, b_1) = (-1)^{\|\rho'\|}(-1)^{|\rho'|}\rho' = -\rho'.$$

• We now treat the cases where $(f^2(x_4, x_3), f^2(x_2, x_1))$ is non zero. The previous calculations show that if $\tilde{\mu}^2(f^2(x_4, x_3), f^2(x_2, x_1))$ is non zero, $\tilde{\mu}^2(f^3(x_4, x_3, x_2), f^1(x_1))$ must be zero.

For (x_3, x_4) going through (j, i, B). Let ρ be a non trivial path in $\mathcal{F}(j, i)$ such that $a_i \rho$ is non zero for $a_i \in \mathcal{A}(i, B)$. Recall that $f^2(a_i, \rho) = \rho$.

On the tuple $(x_1, x_2, x_3, x_4) = (x_1, x_2, \rho, a_i)$, $f^2(x_2, x_1)$ is non zero only if $(x_1, x_2) = (e_k, \gamma)$ where $e_k \in \mathcal{A}(B, k)$ and $\gamma \in \mathcal{F}(k, j)$ decomposes as $\gamma = \gamma' a_k + \gamma' b_k$. In this case $f^2(\gamma, e_k) = -\gamma'$.

Since $f^3(\rho, \gamma, e_k) = 0$, $\Sigma(4) = \tilde{\mu}^2(f^2(x_4, x_3), f^2(x_2, x_1))$.

- (B, k, j, i, B) :

  ▷ $\tilde{\mu}^2(f^2(a_i, \rho), f^2(\gamma, e_k)) = -\tilde{\mu}^2(s^1 \otimes \rho, s^0 \otimes \gamma') = -(-1)^{|\gamma'|} s^1 \otimes \rho\gamma'$. Both g^1 and T^1 vanish on this element.

For (x_3, x_4) going through (B, i, j). Let $\rho \in \mathcal{F}(i, j)$ decomposing as $\rho = \rho' a_i + \rho' b_i$. Recall that $f^2(\rho, e_i) = -\rho'$.

On the tuple $(x_1, x_2, x_3, x_4) = (x_1, x_2, e_i, \rho)$, $f^2(x_2, x_1)$ is non zero in the following two cases.

- (l, k, B, i, j):

  ▷ Let γ be non trivial in $\mathcal{F}(l, k)$ and such that $a_k \gamma$ is non zero for $a_k \in \mathcal{A}(k, B)$. Recall that $f^2(a_k, \gamma) = \gamma$.
  In this case, $f^3(e_i, a_k, \gamma) = 0$ and $\Sigma(4) = \tilde{\mu}^2(f^2(\rho, e_i), f^2(a_k, \gamma)) = -\tilde{\mu}^2(s^0 \otimes \rho', s^1 \otimes \gamma) = 0$.

- (B, k, B, i, j):

  ▷ Recall that $f^2(a_k, e_k) = \sum_{v=1}^{k-1}(e_v + e_{v'}) + e_k$. Since $f^3(e_i, a_k, e_k) = 0$,

$$\Sigma(4) = \tilde{\mu}^2(f^2(\rho, e_i), f^2(a_k, e_k)) = -\tilde{\mu}^2(s^0 \otimes \rho', s^0 \otimes \sum_{v=1}^{k-1}(e_v + e_{v'}) + e_k).$$

  Both g^1 and T^1 vanish on this element.

For (x_3, x_4) going through (B, i, B). Recall that $f^2(a_i, e_i) = \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i$.

On the tuple $(x_1, x_2, x_3, x_4) = (x_1, x_2, e_i, a_i)$, $f^2(x_2, x_1)$ is non zero in the following two cases.

- (j, k, B, i, B):

▷ Let ρ be a non trivial path in $\mathcal{F}(j,k)$ such that $a_k\rho$ is non zero for $a_k \in \mathcal{A}(k,B)$. Recall that $f^2(a_k,\rho)=\rho$. Since $f^3(e_i,a_k,\rho)=0$,

$$\Sigma(4)=\tilde{\mu}^2(f^2(b_i,e_i),f^2(a_k,\rho))=\tilde{\mu}^2(\sum_{v=1}^{i-1}(e_v+e_{v'})+e_i,\rho).$$

Both g^1 and T^1 vanish on this element.

- (B,k,B,i,B):

▷ Recall that $f^2(a_k,e_k)=\sum_{v=1}^{k-1}(e_v+e_{v'})+e_k$. Since $f^3(e_i,a_k,e_k)=0$,

$$\begin{aligned}\Sigma(4)=\tilde{\mu}^2(f^2(a_i,e_i),f^2(a_k,e_k)) &= \tilde{\mu}^2(\sum_{v=1}^{i-1}(e_v+e_{v'})+e_i,\sum_{v=1}^{k-1}(e_v+e_{v'})+e_k)\\ &= \sum_{v=1}^{\min(k,i)-1}(e_v+e_{v'})+e_{\min(k,i)}.\end{aligned}$$

Both g^1 and T^1 vanish on this element.

· We now treat the cases where $(f^1(x_4),f^3(x_3,x_2,x_1))$ is non zero. The previous calculations show that if $\tilde{\mu}^2(f^1(x_4),f^3(x_3,x_2,x_1))$ is non zero, then $\tilde{\mu}^2(f^2(x_4,x_3),f^2(x_2,x_1))$ must be zero. The only case when $\tilde{\mu}^2(f^1(x_4),f^3(x_3,x_2,x_1))$ and $\tilde{\mu}^2(f^3(x_4,x_3,x_2),f^1(x_1))$ are both non zero is when $(x_1,x_2,x_3,x_4)=(b_1,e_1,\rho'b_1,a_k)$.

For (x_1,x_2,x_3) going through (B,i,k,B). Let ρ in $\mathcal{F}(i,k)$ decomposing as $\rho=\rho'a_i+\rho'b_i$, and such that $a_k\rho$ is non zero for $a_k\in\mathcal{A}(k,B)$. Recall that $f^3(a_k,\rho,e_i)=-\rho'$.

- (B,i,k,B,h):

▷ $\tilde{\mu}^2(e_h,f^3(a_k,\rho,e_i))=-\tilde{\mu}^2(s^{-1}\otimes e_h,s^1\otimes\rho')=(-1)^{|\rho'|}s^0\otimes e_h\rho'$. Both g^1 and T^1 vanish on this element.

- (B,i,k,B,B):

▷ $\tilde{\mu}^2(e_B,f^3(a_k,\rho,e_i))=-\tilde{\mu}^2(e_B,\rho')$. Both g^1 and T^1 vanish on this element.

▷ $\tilde{\mu}^2(b_1,f^3(a_k,\rho,e_i))=-\tilde{\mu}^2(b_1,\rho')$. Both g^1 and T^1 vanish on this element.

For (x_1,x_2,x_3) going through $(B,B,1,j)$. Let $\rho'b_1$ in $\mathcal{F}(1,j)$. Recall that $f^3(\rho'b_1,e_1,b_1)=\rho'$.

- $(B,B,1,j,l)$:

▷ For $\gamma\in\mathcal{F}(j,l)$, $\tilde{\mu}^2(\gamma,f^3(\rho'b_1,e_1,b_1))=\tilde{\mu}^2(\gamma,\rho')$. Both g^1 and T^1 vanish on this element.

- $(B,B,1,j,B)$:

Suppose that $j\in A_c$.

▷ The case $(x_1, x_2, x_3, x_4) = (b_1, e_1, \rho' b_1, a_j)$ as already been treated before. See (B, B, i, k, B).

Let $j \in A$ and $i \in A_c$. The non vanishing f^4 are:

- $(B, B, 1, i, B)$: For $\rho' b_1 \in \mathcal{F}(1, i)$ such that $a_i \rho'$ is non zero, $f^4(a_i, \rho' b_1, e_1, b_1) = \rho'$,
- $(B, B, B, 1, j)$: For $\rho' b_1 \in \mathcal{F}(1, j)$, $f^4(\rho' b_1, e_1, b_1, b_1) = -\rho'$.

• $n \geq 5$. We will show by induction that for all $n \geq 5$, $\mu^n = 0$ and that the only non zero f^n are given by:

- $(B, \ldots, B, 1, i, B)$: For $\rho' b_1 \in \mathcal{F}(1, i)$ such that $a_i \rho'$ is non zero,
$$f^n(a_i, \rho' b_1, e_1, b_1, \ldots, b_1) = (-1)^n \rho',$$
- $(B, \ldots, B, B, 1, j)$: For $\rho' b_1 \in \mathcal{F}(1, j)$,
$$f^n(\rho' b_1, e_1, b_1, b_1, \ldots, b_1) = (-1)^{n+1} \rho',$$

for $j \in A$ and $i \in A_c$. It is a direct generalization of the case $n = 3$ and 4.

Let $n \geq 5$ be such that for all $p \in \{4, \ldots, n-1\}$, the only non zero f^p are as above. Each term of $\Sigma(n)$ is of the form $\tilde{\mu}^2(f^{n-s}(x_n, \ldots, x_{s+1}), f^s(x_s, \ldots, x_1))$ with either $s \geq 3$ or $n - s \geq 3$.

If $n - 3 \geq s \geq 3$, then $n - s \geq 3$. Note that it is possible only for $n \geq 6$. Our induction hypothesis ensures that, up to a sign, $f^{n-s}(x_n, \ldots, x_{s+1})$ and $f^s(x_s, \ldots, x_1)$ are of the form:

- $s^1 \otimes \rho' \in \mathcal{A}(B, B)$ for some path $\rho' \in \mathcal{F}(i', k)$, or
- $s^0 \otimes \rho' \in \mathcal{A}(B, j)$ for some path $\rho' \in \mathcal{F}(1, j)$.

Applying $\tilde{\mu}^2$ on a couple of such elements gives zero:

- (B, B, B): $\tilde{\mu}^2(s^1 \otimes \gamma', s^1 \otimes \rho') = 0$,
- (B, B, j): $\tilde{\mu}^2(s^0 \otimes \gamma', s^1 \otimes \rho') = 0$.

Thus

$$\begin{aligned}\Sigma(n) =& \tilde{\mu}^2(f^{n-1}(x_n, \ldots, x_2), f^1(x_1)) + \tilde{\mu}^2(f^{n-2}(x_n, \ldots, x_3), f^2(x_2, x_1)) \\ &+ \tilde{\mu}^2(f^2(x_n, x_{n-1}), f^{n-2}(x_{n-2}, \ldots, x_1)) + \tilde{\mu}^2(f^1(x_n), f^{n-1}(x_{n-1}, \ldots, x_1)).\end{aligned}$$

We compute all the possible values of $\Sigma(n)$. Note that $n - 1 \geq 4$ and $n - 2 \geq 3$. Recall that the non vanishing f^2 are:

- (j, i, B): For $\rho \in \mathcal{F}(j, i)$ non trivial such that $a_i \rho$ is non zero, $f^2(a_i, \rho) = \rho$,
- (B, i, j): For $\rho \in \mathcal{F}(i, j)$ decomposing as $\rho = \rho' a_i + \rho' b_i$ for some ρ', $f^2(\rho, e_i) = -\rho'$,

- (B, i, B): $f^2(a_i, e_i) = \sum_{v=1}^{i-1}(e_v + e_{v'}) + e_i$.

And the non vanishing f^3 are:

- (B, i, k, B): Let ρ in $\mathcal{F}(i, k)$ decomposing as $\rho = \rho' a_i + \rho' b_i$, and such that $a_k \rho$ is non zero for $a_k \in \mathcal{A}(k, B)$. Then $f^3(a_k, \rho, e_i) = -\rho'$,
- $(B, B, 1, j)$: For $\rho' b_1 \in \mathcal{F}(1, j)$, $f^3(\rho' b_1, e_1, b_1) = \rho'$.

Let $j, l \in A$ and $i, k, h \in A_c$.

• Suppose $\tilde{\mu}^2(f^{n-1}(x_n, \dots, x_2), f^1(x_1))$ is non zero. We go through all the possible values of the term $f^{n-1}(x_n, \dots, x_2)$.

For $(x_2, \dots, x_n)$ going through $(B, \dots, B, 1, i, B)$. Let $\rho' b_1 \in \mathcal{F}(1, i)$ such that $a_i \rho'$ is non zero. Recall that $f^{n-1}(a_i, \rho' b_1, e_1, b_1, \dots, b_1) = (-1)^{n-1}\rho'$.

On the tuple $(x_1, x_2, \dots, x_n)$, $f^2(x_2, x_1)$ is necessarily zero since the codomain of x_1 must be B. Similarly $f^{n-2}(x_{n-2}, \dots, x_1)$ is zero since the domain of x_{n-2} must be B. The term $f^{n-1}(x_{n-1}, \dots, x_1)$ is non zero if $x_1 = b_1 \in \mathcal{A}(B, B)$, and in this case recall that $f^{n-1}(\rho' b_1, e_1, b_1, \dots, b_1) = (-1)^n \rho'$.

- $(k, B, \dots, B, 1, i, B)$:
  - ▷ $(-1)^{n-1}\tilde{\mu}^2(s^1 \otimes \rho', s^0 \otimes a_k) = (-1)^{n-1} s^1 \otimes \rho' a_k$. Both g^1 and T^1 vanish on this element.
- $(B, B, \dots, B, 1, i, B)$:
  - ▷ $(-1)^{n-1}\tilde{\mu}^2(s^1 \otimes \rho', e_B) = (-1)^{n-1} s^1 \otimes \rho'$. Both g^1 and T^1 vanish on this element.
  - ▷ First,

$$\begin{aligned}\tilde{\mu}^2(f^{n-1}(a_i, \rho' b_1, e_1, b_1, \dots, b_1), b_1) &= (-1)^{n-1}\tilde{\mu}^2(s^1 \otimes \rho', s^{-1} \otimes b_1) \\ &= (-1)^{n-1}(-1)^{\|\rho'\|} s^0 \otimes \rho' b_1.\end{aligned}$$

    Moreover,

$$\tilde{\mu}^2(a_i, f^{n-1}(\rho' b_1, e_1, b_1, \dots, b_1)) = (-1)^n \tilde{\mu}^2(s^0 \otimes a_i, s^0 \rho') = (-1)^n (-1)^{|\rho'|} s^0 \otimes a_i \rho'.$$

    Thus $\Sigma(n) = (-1)^n(-1)^{|\rho'|}(\rho' b_1 + a_i \rho')$.

    $\to g^1$ vanishes on this element and applying T^1 gives

$$f^n(a_i, \rho' b_1, e_1, b_1, \dots, b_1, b_1) = (-1)^n(-1)^{|\rho'|} T^1(\rho' b_1 + a_i \rho') = (-1)^n \rho'.$$

For $(x_2, \dots, x_n)$ going through $(B, \dots, B, B, 1, j)$. Let $\rho' b_1 \in \mathcal{F}(1, j)$. Recall that

$$f^{n-1}(\rho' b_1, e_1, b_1, b_1, \dots, b_1) = (-1)^n \rho'.$$

On the tuple $(x_1, x_2, \dots, x_n)$, $f^2(x_2, x_1)$ is necessarily zero since the codomain of x_1 must be B. Similarly $f^{n-2}(x_{n-2}, \dots, x_1)$ and $f^{n-1}(x_{n-1}, \dots, x_1)$ are zero since the domain of x_{n-2} and x_{n-1} must be B.

- $(k, B, \ldots, B, B, 1, j)$:
  - ▷ $(-1)^n \tilde{\mu}^2(s^0 \otimes \rho', s^0 \otimes a_k) = (-1)^n s^0 \otimes \rho' a_k = 0.$
- $(B, B, \ldots, B, B, 1, j)$:
  - ▷ Both g^1 and T^1 vanish on $(-1)^n \tilde{\mu}^2(s^0 \otimes \rho', e_B)$.
  - ▷ $(-1)^n \tilde{\mu}^2(s^0 \otimes \rho', s^{-1} \otimes b_1) = (-1)^n (-1)^{\|\rho'\|} s^{-1} \otimes \rho' b_1$.
    $\to g^1$ vanishes on this element and applying T^1 gives

$$f^n(\rho' b_1, e_1, b_1, b_1, \ldots, b_1, b_1) = (-1)^{n+1}(-1)^{|\rho'|} T^1(\rho' b_1) = (-1)^{n+1}\rho'.$$

• Suppose $\tilde{\mu}^2(f^{n-2}(x_n, \ldots, x_3), f^2(x_2, x_1))$ is non zero. The previous computations show that in this case $\tilde{\mu}^2(f^{n-1}(x_n, \ldots, x_2), f^1(x_1))$ must be zero.

Looking at the possible value of $(x_n, \ldots, x_3)$, one can make the following observations. On the tuple $(x_1, x_2, \ldots, x_n)$, the morphism x_2 must go from $h \in A_c$ to B, thus $f^{n-2}(x_{n-2}, \ldots, x_1)$ is zero. Similarly, since the codomain of x_1 must be $h \in A_c$, $f^{n-1}(x_{n-1}, \ldots, x_1)$ is zero.

Recall that $n-2$ is possibly equal to 3.

For $(x_3, \ldots, x_n)$ going through $(B, \ldots, B, i, k, B)$. Let ρ in $\mathcal{F}(i,k)$ decomposing as $\rho = \rho' a_i + \rho' b_i$, and such that $a_k \rho$ is non zero for $a_k \in \mathcal{A}(k, B)$. Recall that

$$f^{n-2}(a_i, \rho, e_i, b_1, \ldots, b_1) = (-1)^{n-2}\rho'.$$

- $(j, h, B, \ldots, B, i, k, B)$:
  - ▷ For $\gamma \in \mathcal{F}(j,h)$ such that $a_h\gamma$ is non zero,

$$(-1)^{n-2}\tilde{\mu}^2(s^1 \otimes \rho', f^2(a_h, \gamma)) = (-1)^{n-2}\tilde{\mu}^2(s^1 \otimes \rho', s^1 \otimes \gamma) = 0.$$

- $(B, h, B, \ldots, B, i, k, B)$:
  - ▷ $(-1)^{n-2}\tilde{\mu}^2(s^1 \otimes \rho', f^2(a_h, e_h)) = (-1)^{n-2}\tilde{\mu}^2(s^1 \otimes \rho', \sum_{v=1}^{h-1}(e_v + e_{v'}) + e_h)$. Both g^1 and T^1 vanish on this element.

For $(x_3, \ldots, x_n)$ going through $(B, \ldots, B, B, 1, j)$. Let $\rho' b_1 \in \mathcal{F}(1,j)$. Recall that

$$f^{n-2}(\rho' b_1, e_1, b_1, b_1, \ldots, b_1) = (-1)^{n-1}\rho'.$$

- $(l, i, B, \ldots, B, B, 1, j)$:
  - ▷ For $\gamma \in \mathcal{F}(l,i)$ such that $a_i\gamma$ is non zero,

$$(-1)^{n-1}\tilde{\mu}^2(s^0 \otimes \rho', f^2(a_i, \gamma)) = (-1)^{n-1}\tilde{\mu}^2(s^0 \otimes \rho', s^1 \otimes \gamma) = 0.$$

- $(B, i, B, \ldots, B, B, 1, j)$:

▷ $(-1)^{n-1}\tilde{\mu}^2(s^0\otimes\rho', f^2(a_i, e_i)) = (-1)^{n-1}\tilde{\mu}^2(s^0\otimes\rho', \sum_{v=1}^{i-1}(e_v+e_{v'})+e_i)$. Both g^1 and T^1 vanish on this element.

• Suppose that $\tilde{\mu}^2(f^2(x_n, x_{n-1}), f^{n-2}(x_{n-2}, \dots, x_1))$ is non zero. The previous computations show that in this case $\tilde{\mu}^2(f^{n-1}(x_n, \dots, x_2), f^1(x_1))$ and $\tilde{\mu}^2(f^{n-2}(x_n, \dots, x_3), f^2(x_2, x_1))$ must be zero.

For $(x_1, \dots, x_{n-2})$ going through $(B, \dots, B, i, k, B)$. Let ρ in $\mathcal{F}(i,k)$ decomposing as $\rho = \rho' a_i + \rho' b_i$, and such that $a_k\rho$ is non zero for $a_k \in \mathcal{A}(k, B)$. Recall that

$$f^{n-2}(a_i, \rho, e_i, b_1, \dots, b_1) = (-1)^{n-2}\rho'.$$

On the tuple $(x_1, x_2, \dots, x_n)$, $f^{n-1}(x_{n-1}, \dots, x_1)$ is zero since the domain of x_{n-1} must be B.

- $(B, \dots, B, i, k, B, h, j)$:

  ▷ For $\gamma \in \mathcal{F}(h, j)$ decomposing as $\gamma = \gamma' a_h + \gamma' b_h$,

$$(-1)^{n-2}\tilde{\mu}^2(f^2(\gamma, e_h), \rho') = -(-1)^{n-2}\tilde{\mu}^2(s^0 \otimes \gamma', s^1 \otimes \rho') = 0.$$

- $(B, \dots, B, i, k, B, h, B)$:

  ▷ $(-1)^{n-2}\tilde{\mu}^2(f^2(a_h, e_h), s^1 \otimes \rho') = (-1)^{n-2}\tilde{\mu}^2(\sum_{v=1}^{h-1}(e_v + e_{v'}) + e_h, s^1 \otimes \rho')$. Both g^1 and T^1 vanish on this element.

For $(x_1, \dots, x_{n-2})$ going through $(B, \dots, B, B, 1, j)$. Let $\rho' b_1 \in \mathcal{F}(1, j)$. Recall that

$$f^{n-2}(\rho' b_1, e_1, b_1, b_1, \dots, b_1) = (-1)^{n-1}\rho'.$$

- $(B, \dots, B, B, 1, j, i, B)$:
  Note that $\tilde{\mu}^2(f^1(x_n), f^{n-1}(x_{n-1}, \dots, x_1))$ must be zero.

  ▷ For $\gamma \in \mathcal{F}(j, i)$ non trivial such that $a_i\gamma$ is non zero,

$$(-1)^{n-1}\tilde{\mu}^2(f^2(a_i, \gamma), s^0 \otimes \rho') = (-1)^{n-1}\tilde{\mu}^2(s^1 \otimes \gamma, s^0 \otimes \rho') = (-1)^{n-1}(-1)^{|\rho'|}s^1 \otimes \gamma\rho'.$$

  Both g^1 and T^1 vanish on this element.

• Suppose that $\tilde{\mu}^2(f^1(x_n), f^{n-1}(x_{n-1}, \dots, x_1))$ is non zero. The previous computations show that in this case $\tilde{\mu}^2(f^{n-2}(x_n, \dots, x_3), f^2(x_2, x_1))$ and $\tilde{\mu}^2(f^2(x_n, x_{n-1}), f^{n-2}(x_{n-2}, \dots, x_1))$ must be zero.

The only possible value of $(x_1, \dots, x_n)$ such that $\tilde{\mu}^2(f^{n-1}(x_n, \dots, x_2), f^1(x_1))$ is non zero is given by $(a_i, \rho' b_1, e_1, b_1, \dots, b_1, b_1)$, going through $(B, B, \dots, B, 1, i, B)$.

For $(x_1, \dots, x_{n-1})$ going through $(B, \dots, B, 1, i, B)$. Let $\rho' b_1 \in \mathcal{F}(1, i)$ such that $a_i\rho'$ is non zero. Recall that $f^{n-1}(a_i, \rho' b_1, e_1, b_1, \dots, b_1) = (-1)^{n-1}\rho'$.

- $(B, \dots, B, 1, i, B, k)$:

▷ $(-1)^{n-1}\tilde{\mu}^2(s^{-1}\otimes e_k, s^1\otimes\rho') = (-1)^n(-1)^{|\rho'|}s^0\otimes e_k\rho'$. Both g^1 and T^1 vanish on this element.

- $(B,\dots,B,1,i,B,B)$:

  ▷ Both g^1 and T^1 vanish on $(-1)^{n-1}\tilde{\mu}^2(e_B, s^1\otimes\rho')$.

  ▷ $(-1)^{n-1}\tilde{\mu}^2(s^{-1}\otimes b_1, s^1\otimes\rho') = 0$.

For $(x_1,\dots,x_{n-1})$ going through $(B,\dots,B,B,1,j)$. Let $\rho' b_1 \in \mathcal{F}(1,j)$. Recall that

$$f^{n-1}(\rho' b_1, e_1, b_1, b_1, \dots, b_1) = (-1)^n\rho'.$$

- $(B,\dots,B,B,1,j,l)$:

  ▷ Let $\gamma \in \mathcal{F}(j,l)$. Both g^1 and T^1 vanish on $(-1)^n\tilde{\mu}^2(s^0\otimes\gamma, s^0\otimes\rho')$.

- $(B,B,\dots,B,1,i,B)$:

  ▷ The case $(x_1,\dots,x_n) = (a_i, \rho' b_1, e_1, b_1, \dots, b_1)$ has already been treated before.

This concludes the induction. Relabeling the basis of $H^*\mathcal{A}$ as in Notations 3.4.7 gives the higher multiplications given in the proposition.

□

3.5 A_∞-quotient of the minimal model and formality

In this section we place ourselves as in Setting 3.4.1. The notion of A_∞-quotient was introduced in [LO06]. We describe here only the case of interest for us, following [HKK17](Section 3.5).

The quotient $\mathcal{D} := \mathcal{D}(H^*\mathcal{A}|B)$ of $H^*\mathcal{A}$ (with the A_∞-structure of Proposition 3.4.8) by the object B is a A_∞-category which has the same objects as $H^*\mathcal{A}$, and whose morphism spaces are given for all $i,j \in \mathcal{A}$ by a decomposition (as vector spaces) $\mathcal{D}(i,j) \simeq \bigoplus_{n\in\mathbb{N}^*} \mathcal{D}^{\langle n\rangle}(i,j)$, where

$\mathcal{D}^{\langle n\rangle}(i,j) = \langle a_n\cdot a_{n-1}\cdot\ldots\cdot a_2\cdot a_1 \mid a_1 \in H^*\mathcal{A}(i,B), a_n \in H^*(B,j)$ and $a_i \in H^*(B,B)$ for $2 \le i \le n-1\rangle$.

Note that for $n=1$, $\mathcal{D}^{\langle 1\rangle}(i,j) = H^*\mathcal{A}(i,j)$. The degrees in $\mathcal{D}$ are

$$|a_n\cdot\ldots\cdot a_1| = |a_n| + \ldots + |a_1| + 1 - n,$$

and thus

$$\|a_n\cdot\ldots\cdot a_1\| = \|a_n\| + \ldots + \|a_1\|.$$

For $r \ge 1$ and $0 = n_0 < n_1 < \cdots < n_r$, the higher multiplications are given by

$$\bar{\mu}^r(a_{n_r}\cdot\ldots a_{n_{r-1}+1}, \dots, a_{n_1}\cdot\ldots\cdot a_1) = \sum_{\substack{j\ge 0\\ 1\le k\le n_1\\ n_r\ge k+j\ge n_{r-1}+1}} (-1)^{\|a_{k-1}\cdot\ldots\cdot a_1\|} a_{n_r}\cdot\ldots\cdot a_{k+j+1}\cdot\mu^{j+1}(a_{k+j},\dots,a_k)\cdot a_{k-1}\cdot\ldots\cdot a_1 \quad (3.7)$$

Since $\bar{\mu}^1(e_B\cdot e_B) = e_B$, the object B becomes isomorphic to zero in homology, and restricting to the full subcategory supported on A gives a quasi-equivalent A_∞-category, that we still denote $\mathcal{D}$.

Setting 3.5.1 *Following notations of Setting 3.4.1, let $\mathcal{D}$ be the full subcategory of the A_∞-quotient $\mathcal{D}(H^*\mathcal{A}|B)$ supported on the set of arcs A, where $H^*\mathcal{A}$ is endowed with the A_∞-structure of Proposition 3.4.8.*

It is a DG category (seen as an A_∞-category). Indeed, let $r \geq 3$. The term μ^{j+1} in Equation 3.7 can be non zero only if $j = 2$. But then $\mu^3(a_{k+2}, a_{k+1}, a_1)$ is zero since a_{k+1} does not have B as domain or codomain. Thus $\bar{\mu}^r = 0$.

3.5.1 Quasi-equivalence with the localization of the topological Fukaya category

Relying on results that will be proved later in this section, we now show that $\mathcal{F}_A(S)$ is Morita equivalent to the A_∞-quotient $\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$, with $\mathcal{F}(\hat{S}) = Tw\mathcal{F}_{\hat{A}}(\hat{S})$ the topological Fukaya category of the smooth marked surface $\hat{S}$ before contraction of the simple closed curve. Here $\mathcal{B}$ denotes the full subcategory supported on objects which are isomorphic to elements in $thick(B)$ after passing to the zero homology.

When $\mathcal{H}$ is a full subcategory of a DG category $\mathcal{G}$, the A_∞-quotient $\mathcal{D}(\mathcal{G}|\mathcal{H})$ coincides with the DG quotient introduced by Drinfeld. Since we are working over a field, the following equivalence [Dri04](Theorem 3.4) tells us that it enhances the triangulated quotient:

$$(\mathcal{G}/\mathcal{H})^{tr} \simeq \mathcal{G}^{tr}/\mathcal{H}^{tr}.$$

For $\mathcal{H}$ a full subcategory of an A_∞-category $\mathcal{G}$, [Sei08](Lemma 3.32) tells us that $\mathcal{H}^{tr}$ is the smallest strictly full triangulated subcategories of $\mathcal{G}^{tr}$ that contains $\mathcal{H}$, which we denote by $\langle\mathcal{H}\rangle$. Thus $(\mathcal{H}^{tr})^\natural$ is $thick(\mathcal{H})$, the smallest strictly full triangulated subcategories of $(\mathcal{G}^{tr})^\natural$ that is closed under direct summands.

Lemma 3.5.2 *Seeing the set of arcs A as objects of the quotient $\mathcal{F}_{\hat{A}}(\hat{S})^{tr}/thick(B)$, one has:*

$$thick(A) = (\mathcal{F}_{\hat{A}}(\hat{S})^{tr}/thick(B))^\natural.$$

Proof: We need to show that the arcs of $\hat{A}\backslash A = \{1', \dots .n'\}$ are in $thick(A)$. For $1 \leq k < n$, let

$$D_k = \bigoplus_{i=k+1}^{n} i[1] \oplus \bigoplus_{i=k}^{n} i'$$

be the twisted complex whose differential $\delta^k = (\delta^k_{i,j})_{i,j}$ is given by $\delta^k_{i',i} = a_i$ and $\delta^k_{(i-1)',i} = b_i$, and zero otherwise. Let $D_n = n'$ be the twisted complex concentrated in degree 0. Recall that the triangulated structure on $\mathcal{F}_{\hat{A}}(\hat{S})^{tr}$ is given by triangles:

$$X \xrightarrow{f} Y \to \mathbf{C}(f) \to X[1],$$

where $\mathbf{C}(f)$ is the mapping cone of the degree zero cocycle f between twisted complexes X and Y. See [Sei08](Equation 3.28) for a definition. As a curve on $\hat{S}$, the twisted complex D_1 corresponds to the Dehn twist of 1 along the simple closed curve γ. Since $\mathbf{C}(1 \xrightarrow{-(b_1+a_1)} D_1)$ is the band object

B associated to γ, $(b_1 + a_1)$ becomes an isomorphism in the Verdier quotient, and D_1 belongs to $thick(A)$. Now suppose that D_{k-1} belongs to $thick(A)$ for $2 \leq k \leq n$. Then

$$\mathbf{C}(k \xrightarrow{-(b_k+a_k)} (k-1)' \oplus D_k) = D_{k-1}$$

shows that $(k-1)'$ and D_k are also in $thick(A)$.

□

Theorem 3.5.3 *Following notations of Setting 3.4.1, there is a Morita equivalence:*

$$\mathcal{F}_A(S) \to \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B}).$$

Proof: The inclusion $\mathcal{D}(\mathcal{F}_{\hat{A}}(\hat{S}) \sqcup B|B) \to \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})$ is a Morita equivalence since

$$\mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B})^{tr} \simeq \mathcal{F}(\hat{S})^{tr}/\mathcal{B}^{tr} \simeq \mathcal{F}(\hat{S})^{tr}/thick(B),$$

and

$$\mathcal{D}(\mathcal{F}_{\hat{A}}(\hat{S}) \sqcup B|B)^{tr} \simeq (\mathcal{F}_{\hat{A}}(\hat{S}) \sqcup B)^{tr}/\langle B \rangle \simeq \mathcal{F}(\hat{S})^{tr}/thick(B).$$

Applying Lemma 3.5.2, we see that the inclusion $\mathcal{D}(\mathcal{A}|B) \to \mathcal{D}(\mathcal{F}_{\hat{A}}(\hat{S}) \sqcup B|B)$ is also a Morita equivalence since $(\mathcal{D}(\mathcal{A}|B)^{tr})^\natural \simeq thick(A)$ in $(\mathcal{D}(\mathcal{F}_{\hat{A}}(\hat{S}) \sqcup B|B)^{tr})^\natural \simeq (\mathcal{F}(\hat{S})^{tr}/thick(B))^\natural$. By construction, there is a quasi-equivalence between $\mathcal{A}$ and $H^*\mathcal{A}$ endowed with the minimal structure of Proposition 3.4.8, and it induces a Morita equivalence between $\mathcal{D}(\mathcal{A}|B)$ and $\mathcal{D}(H^*\mathcal{A}|B)$. As stated in the paragraph before Setting 3.5.1, $\mathcal{D}(H^*\mathcal{A}|B)$ is quasi-equivalent to $\mathcal{D}$. We conclude using Proposition 3.5.10 which assserts that $\mathcal{D}$ is formal, and using Theorem 3.5.11 which tells us that $H^*\mathcal{D} \simeq \mathcal{F}_A(S)$.

□

3.5.2 Formality of the A$_\infty$-quotient and description of its homology

Notations 3.5.4 *Since $H^*\mathcal{A}(i,B)$ and $H^*\mathcal{A}(B,j)$ are zero for $i,j \in A_s$, $\mathcal{D}(i,j) = H^*\mathcal{A}(i,j) = \mathcal{F}(i,j)$ if i or j is in A_s.*

Let $i,j \in A_c$ and $n \geq 2$. Using Notations 3.4.7, a basis for $\mathcal{D}^{\langle n \rangle}(i,j)$ is given by elements of the form:

$$\omega = z_j \cdot \Pi_{k=1}^{p}((e_B\cdot)^{n_k}x\cdot)(e_B\cdot)^{n_0}t_i, \tag{3.8}$$

for some $p \geq 0$ and $n_k \geq 0$ for $k \in \{0,\ldots,p\}$, satisfying $\sum\limits_{k=0}^{p}(1+n_k) = n-1$.

Lemma 3.5.5 *Let $i,j \in A$. The differential $\bar{\mu}^1$ vanishes on $\mathcal{D}^{\langle 1 \rangle}(i,j)$ and for ω as in Equation 3.8,*

$$\bar{\mu}^1(\omega) = \sum_{\substack{l=0 \\ n_l \neq 0 \text{ and even}}}^{p} (-1)^{(n_0+\ldots+n_{l-1})} z_j \cdot \Pi_{k=l+1}^{p}((e_B\cdot)^{n_k}x\cdot)(e_B\cdot)^{n_l-1}\Pi_{k=0}^{l-1}(x\cdot(e_B\cdot)^{n_k})t_i \tag{3.9}$$

Proof: Note that since $\mu^3(a_3, a_2, a_1)$ is zero if a_2 is x or e_B, we have:

$$\bar{\mu}^1(a_n \cdot \ldots \cdot a_1) = \sum_{k=1}^{n-1} (-1)^{\|a_{k-1}\cdot \ldots \cdot a_1\|} a_n \cdot \ldots \cdot \mu^2(a_{k+1}, a_k) \cdot \ldots \cdot a_1$$

For any choice of integers $1 = r_0 < r_1 < \ldots < r_s = n$, this allows us to write:

$$\bar{\mu}^1(a_n \cdot \ldots \cdot a_1) = \sum_{l=0}^{s-1} (-1)^{\|a_{r_l-1}\cdot \ldots \cdot a_1\|} a_n \cdot \ldots \cdot a_{r_{l+1}+1} \cdot \bar{\mu}^1(a_{r_{l+1}} \cdot \ldots \cdot a_{r_l}) \cdot a_{r_l-1} \cdot \ldots \cdot a_1 \qquad (3.10)$$

Let ω be as in Equation 3.8. First we consider the case $p = 0$, that is $\omega = z_j \cdot (e_B\cdot)^{n_0} t_i$.
If $n_0 = 0$, $\bar{\mu}^1(z_j \cdot (e_B\cdot)^{n_0} t_i) = \mu^2(z_j, t_i) = 0$. If $n_0 \geq 1$, since for $k \in \mathbb{N}$, $\|(e_B\cdot)^k t_i\| = -k-1$,

$$\bar{\mu}^1(z_j \cdot (e_B\cdot)^{n_0} t_i) = z_j \cdot (e_B\cdot)^{n_0-1} \mu^2(e_B, t_i) + \sum_{k=0}^{n_0-1} (-1)^{-k-1} z_j \cdot (e_B\cdot)^{n_0-1} t_i$$
$$= (-1)^{|t_i|} z_j \cdot (e_B\cdot)^{n_0-1} t_i + \sum_{k=0}^{n_0-1} (-1)^{-k-1} z_j \cdot (e_B\cdot)^{n_0-1} t_i = \sum_{k=-1}^{n_0-1} (-1)^{-k-1} z_j \cdot (e_B\cdot)^{n_0-1} t_i$$

Thus $\bar{\mu}^1(z_j \cdot (e_B\cdot)^{n_0} t_i)$ is equal to zero if n_0 is odd or zero, and to $z_j \cdot (e_B\cdot)^{n_0-1} t_i$ otherwise, which coincides with the given formula.

Now suppose that $p \geq 1$. Let $s = p+1$ and choose in Equation 3.10 the integers r_l such that $a_n \cdot \ldots \cdot a_{r_p} = z_j \cdot (e_B\cdot)^{n_p} x$ and $a_{r_1} \cdot \ldots \cdot a_1 = x \cdot (e_B\cdot)^{n_0} t_i$, and $a_{r_{l+1}} \cdot \ldots \cdot a_{r_l} = x \cdot (e_B\cdot)^{n_l} x$ for $l \in \{1, \ldots, p-1\}$. Let's study each $\bar{\mu}^1(a_{r_{l+1}} \cdot \ldots \cdot a_{r_l})$.

If $n_l = 0$, $\bar{\mu}^1(x \cdot (e_B\cdot)^{n_l} x) = \mu^2(x, x) = 0$. If $n_l \geq 1$, since for $k \in \mathbb{N}$, $\|(e_B\cdot)^k x\| = -k$,

$$\bar{\mu}^1(x \cdot (e_B\cdot)^{n_l} x) = (-1)^{|x|} x \cdot (e_B\cdot)^{n_l-1} x + \sum_{k=0}^{n_l-1} (-1)^{-k} x \cdot (e_B\cdot)^{n_l-1} x = \sum_{k=-1}^{n_l-1} (-1)^{-k} x \cdot (e_B\cdot)^{n_l-1} x$$

Thus $\bar{\mu}^1(x \cdot (e_B\cdot)^{n_l} x)$ is equal to zero if n_l is odd or zero, and to $-x \cdot (e_B\cdot)^{n_l-1} x$ otherwise.

Similarly one can see that $\bar{\mu}^1(x \cdot (e_B\cdot)^{n_0} t_i)$ is equal to zero if n_0 is odd or zero, and to $x \cdot (e_B\cdot)^{n_0-1} t_i$ otherwise. And that $\bar{\mu}^1(z_j \cdot (e_B\cdot)^{n_p} x)$ is equal to zero if n_p is odd or zero, and to $-z_j \cdot (e_B\cdot)^{n_p-1} x$ otherwise.

Finally Equation 3.10 gives:

$$\bar{\mu}^1(z_j \cdot \Pi_{k=1}^{p}((e_B\cdot)^{n_k} x\cdot)(e_B\cdot)^{n_0} t_i) =$$
$$\sum_{\substack{l=0 \\ n_l \neq 0 \text{ and even}}}^{p} (-1)^{\delta_l} z_j \cdot \Pi_{k=l+1}^{p}((e_B\cdot)^{n_k} x\cdot)(e_B\cdot)^{n_l-1} \Pi_{k=0}^{l-1}(x \cdot (e_B\cdot)^{n_k}) t_i$$

where $\delta_0 = 0$ and $\delta_l = \|\Pi_{k=1}^{l-1}((e_B\cdot)^{n_k} x\cdot)(e_B)^{n_0} t_i\| + 1 = (-n_{l-1} - \ldots - n_0 - 1) + 1$ for $l \in \{1, \ldots, p\}$. □

Notations 3.5.6 *Let (i,j) be a pair of objects in $\mathcal{D}$. For ω as in Equation 3.8, let*

$$c(\omega) = |\{l \in \{0,\dots,p\} \mid n_l \text{ is odd}\}| \text{ and } d(\omega) = |\{l \in \{0,\dots,p\} \mid l \text{ is non zero and even}\}|.$$

Let T^1 be the endomorphism (of graded vector space) of $\mathcal{D}(i,j)$ of degree -1, given on ω by:

$$T^1(\omega) = -\lambda(\omega) \sum_{\substack{l=0 \\ n_l \text{ odd}}}^{p} (-1)^{(n_0+\ldots+n_{l-1})} z_j \cdot \Pi_{k=l+1}^{p}((e_B\cdot)^{n_k} x\cdot)(e_B\cdot)^{n_l+1} \Pi_{k=0}^{l-1}(x\cdot(e_B\cdot)^{n_k}) t_i \tag{3.11}$$

where $\lambda(\omega) = (c(\omega)+d(\omega))^{-1}$ if ω has at least one n_l different from zero, and zero otherwise.
For $\rho \in \mathcal{F}(i,j)$, let $T^1(\rho) = 0$.

Proposition 3.5.7 *Let (i,j) be a pair of objects in $\mathcal{D}$.*
We give a decomposition $\mathcal{D}(i,j) = H^*\mathcal{D}(i,j) \oplus C^*\mathcal{D}(i,j)$, where $H^*\mathcal{D}(i,j)$ is a complement of the image of $\bar{\mu}^1$ in the kernel of $\bar{\mu}^1$, and $C^*\mathcal{D}(i,j)$ is an acyclic complement of $H^*\mathcal{D}(i,j)$.

- *If i or j is in A_s, $H^*\mathcal{D}(i,j) = \mathcal{D}(i,j) = \mathcal{F}(i,j)$, $C^*\mathcal{D}(i,j) = 0$;*
- *For $i,j \in A_c$,*
  $H^*\mathcal{D}(i,j) = \langle z_j \cdot (x\cdot)^p t_i \mid p \in \mathbb{N}\rangle \oplus \mathcal{F}(i,j)$,
  $C^*\mathcal{D}(i,j) = \langle z_j \cdot \Pi_{k=1}^{p}((e_B\cdot)^{n_k} x\cdot)(e_B\cdot)^{n_0} t_i \mid \exists l \in \{0,\dots,p\}, n_l$ is non zero $\rangle$.

Moreover T^1 satisfies Equation 3.5:

$$\bar{\mu}^1 T^1 + T^1 \bar{\mu}^1 = f^1 g^1 - \mathrm{id}, \tag{3.12}$$

where $f^1 : H^*\mathcal{D}(i,j) \to \mathcal{D}(i,j)$ is the inclusion and $g^1 : \mathcal{D}(i,j) \to H^*\mathcal{D}(i,j)$ is the projection with respect to this decomposition.

Proof: By Equation 3.9, $\bar{\mu}^1$ is zero on $H^*\mathcal{D}(i,j)$ and $C^*\mathcal{D}(i,j)$ is a subcomplex of $\mathcal{D}(i,j)$. Equation 3.12 is satisfied on $H^*\mathcal{D}(i,j)$ since T^1 vanishes on these elements. Proving it on $C^*\mathcal{D}(i,j)$ will ensure the acyclicity. The following computation is similar to [CJS23], proof of Theorem 2.5.

Let ω be as in Equation 3.8 with at least one n_l different from zero. In $\bar{\mu}^1 \circ T^1(\omega)$, ω appears with coefficient $-\lambda(\omega)c(\omega)$, and in $T^1 \circ \bar{\mu}^1(\omega)$ with coefficient $-\lambda(\omega)d(\omega)$.

The signs involved in T^1 and μ^1 show that, for any element ω' different from ω, if ω' appears in $\bar{\mu}^1 \circ T^1(\omega')$ with coefficient λ, it will appear in $T^1 \circ \bar{\mu}^1(\omega')$ with coefficient $-\lambda$, and reciprocally.

Thus we have $\bar{\mu}^1 \circ T^1(\omega) + T^1 \circ \bar{\mu}^1(\omega) = -\lambda(\omega)(c(\omega)+d(\omega))\omega = -\omega$.

□

The next lemma will show that $\mathcal{D}$ is a formal A_∞-category.

Notations 3.5.8 *We first rename the basis of $H^*\mathcal{D}$ before describing the multiplication. For $i,k \in A_c$ and $p \in \mathbb{N}$, let ${}_k\beta_i^p = z_k \cdot (x\cdot)^p t_i$, with the convention ${}_k\beta_i = {}_k\beta_i^0$.*

Lemma 3.5.9 *The category $H^*\mathcal{D}$, seen as an A_∞-category, is a sub-A_∞-category of $\mathcal{D}$. Its multiplications are given by:*

- *On morphisms in $\mathcal{F}$, $\bar{\mu}^2$ restricts to $\hat{\mu}^2$,*
- *(j,i,i): For ρ a non trivial path in $\mathcal{F}(j,i)$ such that $a_i\rho$ is non zero, $\bar{\mu}^2({}_i\beta_i,\rho) = (-1)^{\|\rho\|}\rho$,*
- *(i,i,j): For ρa_i a path in $\mathcal{F}(i,j)$, $\bar{\mu}^2(\rho a_i, {}_i\beta_i) = -\rho a_i$,*
- *(i,k,h):*
  - ▷ *If $p,q=0$,*
    - *If $i<h$*
      - *If $i<k<h$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i) = -{}_h\beta_i^1 + {}_h\beta_i$,*
      - *Otherwise $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i) = -{}_h\beta_i^1$,*
    - *If $h \le i$*
      - *If $h\le k\le i$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i) = -{}_h\beta_i^1 - {}_h\beta_i$,*
      - *Otherwise $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i) = -{}_h\beta_i^1$,*
  - ▷ *If $p=0$ and $q\ge 1$,*
    - *If $k\le i$, $\bar{\mu}^2({}_h\beta_k^q, {}_k\beta_i) = -{}_h\beta_i^{q+1} - {}_h\beta_i^q$,*
    - *If $i<k$, $\bar{\mu}^2({}_h\beta_k^q, {}_k\beta_i) = -{}_h\beta_i^{q+1}$,*
  - ▷ *If $q=0$ and $p\ge 1$,*
    - *If $k<h$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i^p) = -{}_h\beta_i^{p+1}$,*
    - *If $h\le k$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i^p) = -{}_h\beta_i^{p+1} - {}_h\beta_i^p$,*
  - ▷ *If $p,q\ge 1$, $\bar{\mu}^2({}_h\beta_k^q, {}_k\beta_i^p) = -{}_h\beta_i^{q+p+1} - {}_h\beta_i^{q+p}$.*

The elements $e_1,\dots,e_n$ are units.

Proof: We compute $\bar{\mu}^2$ on all possible couples (x_1,x_2) of basis elements.
Let $j,l,m\in A$, $i,k,h\in A_c$, and $p,q\in\mathbb{N}$.

- (j,l,m): Let $(x_1,x_2)=(\rho,\gamma)$, where $\rho\in\mathcal{F}(j,l)$ and $\gamma\in\mathcal{F}(l,m)$ are paths.
$$\bar{\mu}^2(\gamma,\rho) = \hat{\mu}^2(\gamma,\rho).$$
- (j,i,k): Let $(x_1,x_2)=(\rho, z_k\cdot(x\cdot)^p t_i)$, where ρ is a path in $\mathcal{F}(j,i)$.
If $j=i$ and $\rho=e_i$, $\bar{\mu}^2(z_k\cdot(x\cdot)^p t_i, e_i) = z_k\cdot(x\cdot)^p\mu^2(t_i,e_i) = z_k\cdot(x\cdot)^p t_i$.
Otherwise the term $\bar{\mu}^2(z_k\cdot(x\cdot)^p t_i,\rho)$ is non zero only for $p=0$ and $i=k$, and if ρ is non trivial such that $a_i\rho$ is non zero. In this case,
$$\bar{\mu}^2(z_i\cdot t_i,\rho) = \mu^3(z_i,t_i,\rho) = (-1)^{\|\rho\|}\rho.$$
- (i,k,j): Let $(x_1,x_2)=(z_k\cdot(x\cdot)^p t_i,\rho)$, where ρ is a path in $\mathcal{F}(k,j)$.
If $j=k$ and $\rho=e_k$, $\bar{\mu}^2(e_k, z_k\cdot(x\cdot)^p t_i) = (-1)^{\|(x\cdot)^p t_i\|}\mu^2(e_k,z_k)\cdot(x\cdot)^p t_i = z_k\cdot(x\cdot)^p t_i$.
Otherwise the term $\bar{\mu}^2(\rho, z_k\cdot(x\cdot)^p t_i)$ is non zero only for $p=0$ and $i=k$, and if ρ it factors as $\rho=\rho' a_i$. In this case,
$$\bar{\mu}^2(\rho' a_i, z_i\cdot t_i) = \mu^3(\rho, z_i, t_i) = -\rho' a_i.$$

- (i,k,h): Let $(x_1,x_2)=(z_k\cdot(x\cdot)^p t_i, z_h\cdot(x\cdot)^q t_k)$.
  We compute the different possible values of $\bar{\mu}^2(z_h\cdot(x\cdot)^q t_k, z_k\cdot(x\cdot)^p t_i)$.

  ▷ If $p,q=0$,

  $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot t_i)=(-1)^{\|t_i\|}z_h\cdot\mu^2(t_k,z_k)\cdot t_i+z_h\cdot\mu^3(t_k,z_k,t_i)+(-1)^{\|t_i\|}\mu^3(z_h,t_k,z_k)\cdot t_i$$

  - If $i>k-1$,
    $\rightarrow$ If $h>k$,
    $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot t_i)=(-1)^{\|t_i\|}z_h\cdot\mu^2(t_k,z_k)\cdot t_i=-z_h\cdot x\cdot t_i$$
    $\rightarrow$ If $1\leq h\leq k$,
    $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot t_i)=-z_h\cdot x\cdot t_i-z_h\cdot t_i$$
  - If $1\leq i\leq k-1$,
    $\rightarrow$ If $h>k$,
    $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot t_i)=-z_h\cdot x\cdot t_i+z_h\cdot t_i$$
    $\rightarrow$ If $1\leq h\leq k$,
    $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot t_i)=-z_h\cdot x\cdot t_i+z_h\cdot t_i-z_h\cdot t_i=-z_h\cdot x\cdot t_i$$

  ▷ If $p=0$ and $q\geq 1$,

  $$\begin{aligned}&\bar{\mu}^2(z_h\cdot(x\cdot)^q t_k, z_k\cdot t_i)\\ =&(-1)^{\|t_i\|}z_h\cdot(x\cdot)^q\mu^2(t_k,z_k)\cdot t_i+z_h\cdot(x\cdot)^q\mu^3(t_k,z_k,t_i)+(-1)^{\|t_i\|}z_h\cdot(x\cdot)^{q-1}\mu^3(x,t_k,z_k)\cdot t_i\\ =&-z_h\cdot(x\cdot)^{q+1}\cdot t_i+z_h\cdot(x\cdot)^q\mu^3(t_k,z_k,t_i)-z_h\cdot(x\cdot)^q\cdot t_i\end{aligned}$$

  - If $i>k-1$,
    $$\bar{\mu}^2(z_h\cdot(x\cdot)^q t_k, z_k\cdot t_i)=-z_h\cdot(x\cdot)^{q+1}\cdot t_i-z_h\cdot(x\cdot)^q\cdot t_i$$
  - If $1\leq i\leq k-1$,
    $$\begin{aligned}\bar{\mu}^2(z_h\cdot(x\cdot)^q t_k, z_k\cdot t_i)&=-z_h\cdot(x\cdot)^{q+1}\cdot t_i+z_h\cdot(x\cdot)^q t_i-z_h\cdot(x\cdot)^q\cdot t_i\\ &=-z_h\cdot(x\cdot)^{q+1}\cdot t_i\end{aligned}$$

  ▷ If $q=0$ and $p\geq 1$,

  $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot(x\cdot)^p t_i)=(-1)^{\|(x\cdot)^p t_i\|}z_h\cdot\mu^2(t_k,z_k)\cdot(x\cdot)^p t_i+(-1)^{\|(x\cdot)^p t_i\|}\mu^3(z_h,t_k,z_k)\cdot(x\cdot)^p t_i$$

  - If $h>k$,
    $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot(x\cdot)^p t_i)=-z_h\cdot(x\cdot)^{p+1}t_i$$
  - If $1\leq h\leq k$,
    $$\bar{\mu}^2(z_h\cdot t_k, z_k\cdot(x\cdot)^p t_i)=-z_h\cdot(x\cdot)^{p+1}t_i-z_h\cdot(x\cdot)^p t_i$$

▷ If $p, q \geq 1$,

$$\begin{aligned}&\bar{\mu}^2(z_h \cdot (x\cdot)^q t_k, z_k \cdot (x\cdot)^p t_i)\\ &\quad = (-1)^{\|(x\cdot)^p t_i\|} z_h \cdot (x\cdot)^q \mu^2(t_k, z_k) \cdot (x\cdot)^p t_i + (-1)^{\|(x\cdot)^p t_i\|} z_h \cdot (x\cdot)^{q-1} \mu^3(x, t_k, z_k) \cdot (x\cdot)^p t_i\\ &\quad = -z_h \cdot (x\cdot)^{q+1+p} t_i - z_h \cdot (x\cdot)^{q+p} t_i\end{aligned}$$

Relabeling the basis of $H^*\mathcal{D}$ as in Notations 3.5.8 gives:

- (i, k, h):
  - ▷ If $p, q = 0$,
    - If $i > k-1$ and $1 \leq h \leq k$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i) = -{}_h\beta_i^1 - {}_h\beta_i$
    - If $1 \leq i \leq k-1$ and $h > k$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i) = -{}_h\beta_i^1 + {}_h\beta_i$,
    - Otherwise, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i) = -{}_h\beta_i^1$,
  - ▷ If $p = 0$ and $q \geq 1$,
    - If $i > k-1$, $\bar{\mu}^2({}_h\beta_k^q, {}_k\beta_i) = -{}_h\beta_i^{q+1} - {}_h\beta_i^q$,
    - If $1 \leq i \leq k-1$, $\bar{\mu}^2({}_h\beta_k^q, {}_k\beta_i) = -{}_h\beta_i^{q+1}$,
  - ▷ If $q = 0$ and $p \geq 1$,
    - If $h > k$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i^p) = -{}_h\beta_i^{p+1}$,
    - If $1 \leq h \leq k$, $\bar{\mu}^2({}_h\beta_k, {}_k\beta_i^p) = -{}_h\beta_i^{p+1} - {}_h\beta_i^p$,
  - ▷ If $p, q \geq 1$, $\bar{\mu}^2({}_h\beta_k^q, {}_k\beta_i^p) = -{}_h\beta_i^{q+1+p} - {}_h\beta_i^{q+p}$.

Finally, we reorganise the case disjunction.

□

In this situation, applying the minimal model construction on $H^*\mathcal{D}$, seen as a sub-A_∞-category of $\mathcal{D}$, will give the restriction of $\bar{\mu}$ as A_∞-structure, and the quasi-equivalence will be given by the inclusion. This gives the next proposition.

Proposition 3.5.10 *The A_∞-category $\mathcal{D}$ is formal, and the inclusion $f^1 : H^*\mathcal{D} \to \mathcal{D}$ is a quasi-equivalence.*

The rest of this section will be devoted to prove the following theorem, which gives a description of $H^*\mathcal{D}$ (seen as a category) as the path category over a quiver with relations.

Theorem 3.5.11 *Following notations of Setting 3.4.1 and Setting 3.5.1, there is an equivalence of categories:*

$$H^*\mathcal{D} \simeq \mathcal{F}_A(S).$$

Recall that $\mathcal{F}_A(S)$ is defined in Definition 3.3.6 as $\mathcal{P}(Q, I)$, where (Q, I) is the graded pinched gentle bound quiver associated to the dissection A. The proof will rely on an explicit description, done in Subsection 3.5.3, of the morphism spaces of $\mathcal{P}(Q, I)$. We first describe further the $\mathbb{Z}$-graded category $H^*\mathcal{D}$. Note that passing from $H^*\mathcal{D}$ seen as an A_∞-category to $H^*\mathcal{D}$ seen as a category, by changing the signs, does not affect the multiplication given in Lemma 3.5.9 outisde of $\mathcal{F}$ since $|{}_k\alpha_i| = 0$. On these elements, we denote $\bar{\mu}^2(y, x)$ by yx.

For $i, k \in \{1, \ldots, n\}$, let ${}_k\alpha_i = 2{}_k\beta_i$ and $\gamma_i^p = (2{}_i\beta_i + e_i)^p$ for all $p \in \mathbb{N}$, with the convention $\gamma_i^0 = e_i$. Moreover, let ${}_k\alpha_i^+ = {}_k\alpha_{k-1} \ldots {}_{i+1}\alpha_i$ and ${}_k\alpha_i^- = {}_k\alpha_{k+1} \ldots {}_{i-1}\alpha_i$, where no ${}_{l\pm1}\alpha_l$ is repeated.

Lemma 3.5.12 *Let $i \neq k$ in $\{1, \dots, n\}$. The following relations hold.*

$$ {}_k\alpha_i\gamma_i = \gamma_{kk}\alpha_i \tag{3.13}$$

$$ {}_i\alpha_{kk}\alpha_i = \gamma_i^2 - e_i \tag{3.14}$$

$$ {}_k\alpha_i^+ = {}_k\alpha_i(\gamma_i + e_i)^{l^+} \tag{3.15}$$

where l^+ is the length of ${}_k\alpha_{k-1} \dots {}_{i+1}\alpha_i$ minus one: $l = k - i - 1$ if $i < k$ and $l = n - i + k - 1$ otherwise.

$$ {}_k\alpha_i^- = {}_k\alpha_i(\gamma_i - e_i)^{l^-} \tag{3.16}$$

where l^- is the length of ${}_k\alpha_{k+1} \dots {}_{i-1}\alpha_i$ minus one: $l = n - k + i - 1$ if $i < k$ and $l = i - k - 1$ otherwise.

$$\begin{aligned} {}_i\alpha_i^+ &= (\gamma_i - e_i)(\gamma_i + e_i)^{n-1} \\ {}_i\alpha_i^- &= (\gamma_i + e_i)(\gamma_i - e_i)^{n-1} \end{aligned} \tag{3.17}$$

Note that l^+ and l^- always satisfy $l^+ + l^- = n - 2$.
Proof:

(i) First, $2_k\beta_i(2_i\beta_i + e_i) = 4_k\beta_{ii}\beta_i + 2_k\beta_i$ and $2(2_k\beta_k + e_k)_k\beta_i = 4_k\beta_{kk}\beta_i + 2_k\beta_i$.
If $i < k$, ${}_k\beta_{ii}\beta_i = -{}_k\beta_i^1 = {}_k\beta_{kk}\beta_i$.
If $k \leq i$, ${}_k\beta_{ii}\beta_i = -{}_k\beta_i^1 - {}_k\beta_i = {}_k\beta_{kk}\beta_i$,

(ii) First, $4_i\beta_{kk}\beta_i = -4_i\beta_i^1$. Then

$$(2_i\beta_i + e_i)(2_i\beta_i + e_i) = 4_i\beta_{ii}\beta_i + 4_i\beta_i + e_i = -4({}_i\beta_i^1 + {}_i\beta_i) + 4_i\beta_i + e_i = -4_i\beta_i + e_i.$$

(iii) (a) Suppose $i < k$. It is enough to show that for all $i < h < k$, $4_k\beta_{hh}\beta_i = 2_k\beta_i(\gamma_i + e_i)$.
First $4_k\beta_{hh}\beta_i = -4_k\beta_i^1 + 4_k\beta_i$. Then

$$2_k\beta_i(2_i\beta_i + 2e_i) = 4_k\beta_{ii}\beta_i + 4_k\beta_i = -4_k\beta_i^1 + 4_k\beta_i.$$

Similarly if $i > k$, we show that for all $h < k$ and $i < h$, $4_k\beta_{hh}\beta_i = 2_k\beta_i(\gamma_i + e_i)$.
First $4_k\beta_{hh}\beta_i = -4_k\beta_i^1$. Then

$$2_k\beta_i(\gamma_i + e_i) = 4_k\beta_{ii}\beta_i + 4_k\beta_i = -4({}_k\beta_i^1 + {}_k\beta_i) + 4_k\beta_i = -4_k\beta_i^1.$$

(b) Suppose $i < k$. It is enough to show that for all $h < i$ and $k < h$, $4_k\beta_{hh}\beta_i = 2_k\beta_i(\gamma_i - e_i)$.
First $4_k\beta_{hh}\beta_i = -4_k\beta_i^1$. Then

$$2_k\beta_i(2_i\beta_i + e_i - e_i) = 4_k\beta_{ii}\beta_i = -4_k\beta_i^1.$$

Similarly if $i > k$, we show that for $k < h < i$, $4_k\beta_{hh}\beta_i = 2_k\beta_i(\gamma_i - e_i)$.
First $4_k\beta_{hh}\beta_i = -4_k\beta_i^1 - 4_k\beta_i$. Then

$$2_k\beta_i(\gamma_i - e_i) = 4_k\beta_{ii}\beta_i = -4_k\beta_i^1 - 4_k\beta_i.$$

(iv) By the last item, we have ${}_i\alpha_{i-1\,i-1}\alpha_{i-2}\ldots{}_{i+1}\alpha_i = {}_i\alpha_{i-1\,i-1}\alpha_i(\gamma_i+e_i)^{n-2}$ which in turn is equal to $(\gamma_i^2-e_i)(\gamma_i+e_i)^{n-2}$ by the second item. The other direction is similar.

$\square$

Lemma 3.5.13 *Let $i \neq k \in \{1,\ldots,n\}$. Let $B_{i,k}$ be a basis of $\mathcal{F}(i,k)$, and B_i a basis of a complement of $\langle e_i\rangle$ in $\mathcal{F}(i,i)$.*

(1) $\{\gamma_i^p \mid p\in\mathbb{N}\}\sqcup B_i$ is a basis of $H^*\mathcal{D}(i,i)$,

(2) $\{{}_k\alpha_i\gamma_i^p \mid p\in\mathbb{N}\}\sqcup B_{i,k}$ is a basis of $H^*\mathcal{D}(i,k)$.

Proof: (1) Let ${}_i\beta_i^{-1} = e_i$. A basis of $H^*\mathcal{D}(i,i)$ is given by $\{{}_i\beta_i^p \mid p\in\mathbb{Z}_{\geq -1}\}\sqcup B_i$. We verify by induction on $p\in\mathbb{N}$ that $\gamma_i^p = \sum\limits_{s=-1}^{p-1}\lambda_s^p{}_i\beta_i^s$ with $\lambda_{p-1}^p\neq 0$, which ensures that $\{\gamma_i^p\mid p\in\mathbb{N}\}\sqcup B_i$ is linearly independent and spans $H^*\mathcal{D}(i,i)$. We have

$$\gamma_i^{p+1} = (2{}_i\beta_i+e_i)\sum_{s=-1}^{p-1}\lambda_s^p{}_i\beta_i^s = \sum_{s=-1}^{p-1}\lambda_s^p 2{}_i\beta_{i\,i}\beta_i^s + \sum_{s=-1}^{p-1}\lambda_s^p{}_i\beta_i^s$$

with ${}_i\beta_{i\,i}\beta_i^{-1} = {}_i\beta_i$ and ${}_i\beta_{i\,i}\beta_i^s = -{}_i\beta_i^{s+1} - {}_i\beta_i^s$ for $s\geq 0$.

(2) A basis of $H^*\mathcal{D}(i,k)$ is given by $\{{}_k\beta_i^p\mid p\in\mathbb{N}\}\sqcup B_{i,k}$. Similarly, it is enough verify by induction on $p\in\mathbb{N}$ that ${}_k\alpha_i\gamma_i^p = \sum\limits_{s=0}^{p}\lambda_s^p{}_k\beta_i^s$ with $\lambda_p^p\neq 0$. We have

$${}_k\alpha_i\gamma_i^{p+1} = (\sum_{s=0}^{p}\lambda_s^p{}_k\beta_i^s)(2{}_i\beta_i+e_i) = \sum_{s=0}^{p}\lambda_s^p 2{}_k\beta_i^s{}_i\beta_i + \sum_{s=0}^{p}\lambda_s^p{}_k\beta_i^s$$

with ${}_k\beta_{i\,i}\beta_i = -{}_k\beta_i^1$ or $-{}_k\beta_i^1 - {}_k\beta_i$, and ${}_k\beta_i^s{}_i\beta_i = -{}_k\beta_i^{s+1} - {}_k\beta_i^s$.

$\square$

Remark 3.5.14 *Let $i\neq k\in\{1,\ldots,n\}$. Since $(X+1)$ and $(X-1)$ are relatively prime, the following is also a basis of $H^*\mathcal{D}(i,k)$:*

$$\{\big({}_k\alpha_i^+U_{l^+}(\gamma_i) + {}_k\alpha_i^-V_{l^-}(\gamma_i)\big)\gamma_i^p\mid p\in\mathbb{N}\},$$

where for $a,b\in\mathbb{N}$, $U_a,V_b\in K[X]$ are such that $(X+1)^aU_a+(X-1)^bV_b=1$. Using Equations 3.15 and 3.16, this comes from

$$\begin{aligned}{}_k\alpha_i &= {}_k\alpha_i(\gamma_i+e_i)^{l^+}U_{l^+}(\gamma_i) + {}_k\alpha_i(\gamma_i-e_i)^{l^-}V_{l^-}(\gamma_i)\\ &= {}_k\alpha_i^+U_{l^+}(\gamma_i) + {}_k\alpha_i^-V_{l^-}(\gamma_i).\end{aligned}$$

We now prove the theorem, relying on Proposition 3.5.15.

Proof of Theorem 3.5.11: Let $\psi:\mathcal{P}(\overline{Q},\overline{I})\to H^*\mathcal{D}$ be the map given by the identity on arrows. By Lemma 3.5.12 and the first three points of Lemma 3.5.9, it is well defined. It is dense by definition, so it only remains to show that it is fully faithful. Lemma 3.5.13 and Remark 3.5.14 tell us that for

$i, k \in \{1, \ldots, n\}$, the image of the basis of Proposition 3.5.15 spans $H^*\mathcal{D}(i,k)$. It is immediate that it forms a basis for $n \leq 2$, so we now suppose $n \geq 3$. Let us show that it is linearly independant when $i \neq k$. By Equations 3.15 and 3.16,

$$_k\alpha_i^+\gamma_i^p = {}_k\alpha_i(\gamma_i + e_i)^{l^+}\gamma_i^p = {}_k\alpha_i \sum_{s=0}^{l^+} \binom{l^+}{s} \gamma_i^{s+p},$$

$$\text{and } {}_k\alpha_i^-\gamma_i^p = {}_k\alpha_i(\gamma_i - e_i)^{l^-}\gamma_i^p = {}_k\alpha_i \sum_{s=0}^{l^-} \binom{l^-}{s} (-1)^{l^- - s}\gamma_i^{s+p}$$

Given a finite linear combinaison in the image of the basis of Proposition 3.5.15 which is equal to zero, we can successively eliminate the terms $_k\alpha_i^+\gamma_i^p$ for p maximal and greater or equal than l^-. Thus it remains to show that the following family is linearly independent:

$$V_{l^-} = \{{}_k\alpha_i^+\gamma_i^p \mid p < l^-\} \sqcup \{{}_k\alpha_i^-\gamma_i^p \mid p < l^+\}$$

For a fixed n, we will show this by induction on $l^- \in \{0, \ldots, n-2\}$. In what follows, we identify $H^*\mathcal{D}(i,k)$ with the polynomial ring $K[\gamma_i]$, and polynomials in γ_i with their corresponding coordinate vectors in the canonical basis. Let A_{l^-} be the square matrix of size $n-2$ whose first l^- columns are $(\gamma_i + e_i)^{l^+}, \ldots, (\gamma_i + e_i)^{l^+}\gamma_i^{l^- - 1}$, and whose last l^+ columns are $(\gamma_i - e_i)^{l^-}, \ldots, (\gamma_i - e_i)^{l^-}\gamma_i^{l^+ - 1}$.

The case $l^- = 0$ is trivial, as well as $l^- = n-2$. Suppose that $l^- \in \{0, \ldots, n-4\}$ is such that V_{l^-} is linearly independent. We can perform the following column operations on A_{l^-} to obtain a new basis W_{l^-} of $\langle V_{l^-} \rangle$:

$$(\gamma_i - e_i)^{l^-}\gamma_i^j \longleftarrow (\gamma_i - e_i)^{l^-}\gamma_i^{j+1} - (\gamma_i - e_i)^{l^-}\gamma_i^j = (\gamma_i - e_i)^{l^- + 1}\gamma_i^j$$

for $j \in \{0, \ldots, l^+ - 2\}$, where the left arrow indicates a column replacement. Similarly we apply the following transformation to $A_{l^- + 1}$:

$$(\gamma_i + e_i)^{l^+ - 1}\gamma_i^j \longleftarrow (\gamma_i + e_i)^{l^+ - 1}\gamma_i^{j+1} + (\gamma_i + e_i)^{l^+ - 1}\gamma_i^j = (\gamma_i + e_i)^{l^+}\gamma_i^j$$

for $j \in \{0, \ldots, (l^- + 1) - 2\}$. After transformation, the two matrices differ only by one column. The column $(\gamma_i - e_i)^{l^-}\gamma_i^{l^+ - 1}$ in A_{l^-} and $(\gamma_i + e_i)^{l^+ - 1}\gamma_i^{l^-}$ in $A_{l^- + 1}$. Our induction hypothesis tells us that the common columns are linearly independant, and we want to show that it remains the case when adding $(\gamma_i + e_i)^{l^+}\gamma_i^{l^-}$. This is equivalent to showing that the decomposition of $(\gamma_i + e_i)^{l^+ - 1}\gamma_i^{l^-}$ on the basis W_{l^-} involves a non-zero coefficient for $(\gamma_i - e_i)^{l^-}\gamma_i^{l^+ - 1}$. Let

$$(\gamma_i + e_i)^{l^+ - 1}\gamma_i^{l^-} = \sum_{j=0}^{l^- - 1} \lambda_j (\gamma_i + e_i)^{l^+}\gamma_i^j + \sum_{j=0}^{l^+ - 2} \mu_j (\gamma_i - e_i)^{l^- + 1}\gamma_i^j - \mu_{l^+ - 1}(\gamma_i - e_i)^{l^-}\gamma_i^{l^+ - 1}$$

be this decomposition. It can be rewritten as

$$\begin{aligned}
(\gamma_i + e_i)^{l^+ - 1}\gamma_i^{l^-} - \sum_{j=0}^{l^- - 1} \lambda_j \big((\gamma_i + e_i)^{l^+ - 1}\gamma_i^{j+1} + (\gamma_i + e_i)^{l^+ - 1}\gamma_i^j\big) \\
= \sum_{j=0}^{l^+ - 2} \mu_j \big((\gamma_i - e_i)^{l^-}\gamma_i^{j+1} - (\gamma_i - e_i)^{l^-}\gamma_i^j\big) - \mu_{l^+ - 1}(\gamma_i - e_i)^{l^-}\gamma_i^{l^+ - 1} \\
\Leftrightarrow (\gamma_i + e_i)^{l^+ - 1} P(\gamma_i) = (\gamma_i - e_i)^{l^-} Q(\gamma_i)
\end{aligned} \tag{3.18}$$

where $P(\gamma_i) = \sum_{j=0}^{l^-} -(\lambda_j + \lambda_{j-1})\gamma_i^j$ with $\lambda_{l^-} = -1$ and $\lambda_{-1} = 0$, and $Q(\gamma_i) = \sum_{j=0}^{l^+-1} (\mu_{j-1} - \mu_j)\gamma_i^j$ with $\mu_{-1} = 0$. Since W_{l^-} is a basis, there is a unique P of degree l^- and Q of degree $l^+ - 1$ satisfying Equation 3.18. The condition $\mu_{l^+-1} \neq 0$ is equivalent to $Q(e_i) \neq 0$. We can see that it holds by noticing that $P = (\gamma_i - e_i)^{l^-}$ and $Q = (\gamma_i + e_i)^{l^+-1}$.

□

3.5.3 Basis for pinched gentle algebras

Throughout this subsection, let $\Lambda = KQ/\langle I \rangle$ be a pinched gentle algebra, as defined in 3.3.2. In particular, it comes with a collection of disjoint ordered sets of vertices $C_1, \dots, C_r \subseteq Q_0$ called cycles. We will give a basis of Λ using Bergman's diamond lemma [Ber78]. As pointed out in Remark 3.3.4, for all vertices i and k in Q_0 that does not belong to a same cycle C_j, one has an isomorphism of vector spaces $e_k \Lambda e_i \simeq e_k \Lambda^g e_i$. Thus we only need to describe $e_k \Lambda e_i$ for i and k belonging to a common cycle. Let C be a cycle of Λ, which we suppose identified with $\{1, \dots, n\}$.

Proposition 3.5.15 *Let $i \neq k \in C = \{1, \dots, n\}$.*

(1) $\{\gamma_i^p \mid p \in \mathbb{N}^*\} \sqcup B_i$ is a basis of $e_i \Lambda e_i$, where B_i is a basis of $e_i \Lambda^g e_i$,

(2) $\{{}_k\alpha_i^+ \gamma_i^p \mid p \in \mathbb{N}\} \sqcup \{{}_k\alpha_i^- \gamma_i^p \mid p < l^+\} \sqcup B_{i,k}$ is a basis of $e_k \Lambda e_i$, where $B_{i,k}$ is a basis of $e_k \Lambda^g e_i$.

We first recall Bergman's diamond lemma following [BW23]. For Q a quiver, we denote by Q_n the set of path of length $n \in \mathbb{N}$ and $Q_{\geq n_0} = \bigcup_{n \geq n_0} Q_n$ the set of paths of length greater or equal than n_0.

Definition 3.5.16 *Let Q be a finite quiver. A* reduction system *for KQ is a set:*

$$R = \{(s, \varphi_s) \mid s \in S \text{ and } \varphi_s \in KQ\}$$

where

- *S is a subset of $Q_{\geq 2}$ such that for all $s \in S$, no $s' \neq s$ in S is a subpath of s,*
- *For all $s \in S$, s and φ_s are parallel,*
- *For each $(s, \varphi_s) \in R$, φ_s is a linear combination of paths that does not contains elements of S as subpath.*

Paths as in the last item are called irreducible*, and in this case φ_s itself is also called irreducible. The set of all irreducible paths is denoted $\mathrm{Irr}_S(Q)$.*

Definition 3.5.17 *Let R be a reduction system for KQ.*

- *For $(s, \varphi_s) \in R$ and $q, r \in Q_{\geq 0}$ such that $qsr \neq 0$, the basic reduction $\mathfrak{r}_{q,s,r} : KQ \to KQ$ is the linear map defined by $\mathfrak{r}_{q,s,r}(qsr) = q\varphi_s r$ and $\mathfrak{r}_{q,s,r}(p) = p$ for paths $p \neq qsr$. A basic reduction of the form $\mathfrak{r}_{q,s,r}$ is said to be a reduction of type s. A* reduction *is a composition of basic reductions.*

- *A path p is said to be:*
  - reduction-finite *if for any infinite sequence of reductions $(\mathfrak{r}_j)_{j\in\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\mathfrak{r}_n \circ \ldots \circ \mathfrak{r}_0(p) = \mathfrak{r}_{n_0} \circ \ldots \circ \mathfrak{r}_0(p)$,*
  - reduction-unique *if it is reduction-finite and for all reductions $\mathfrak{r}, \mathfrak{r}'$ such that $\mathfrak{r}(p)$ and $\mathfrak{r}'(p)$ are both irreducible, $\mathfrak{r}(p) = \mathfrak{r}'(p)$.*

  R itself is reduction-finite (resp. reduction-unique) if all paths are reduction-finite (resp. reduction-unique).
- *R is said to satisfy the* condition $(\diamond)$ *for the two-sided ideal $I = \langle s - \varphi_s \mid (s, \varphi_s) \in R\rangle$ if it is reduction-unique.*
- *An* overlap ambiguity *of R is a path pqr with $p, q, r \in Q_{\geq 1}$ and $pq, qr \in S$. An overlap ambiguity pqr is* resolvable *if $\varphi_{pq}r$ and $p\varphi_{qr}$ are reduction-finite and there exists reductions $\mathfrak{r}, \mathfrak{r}'$ such that $\mathfrak{r}(\varphi_{pq}r) = \mathfrak{r}'(p\varphi_{qr})$.*

Proposition 3.5.18 [Ber78](Theorem 1.2) *Let $R = \{(s, \varphi_s)\}$ be a reduction system for the path algebra KQ and let $I = \langle s - \varphi_s \mid (s, \varphi_s) \in R\rangle$ be the corresponding two-sided ideal. If R is reduction-finite, then the following are equivalent:*

(i) All overlap ambiguities of R are resolvable,

(ii) R is reduction-unique, ie. R satisfies $(\diamond)$ for I,

(iii) The image of $Irr_S(Q)$ under $\pi \to KQ/I$ is a K-basis of kQ/I.

Proof of Proposition 3.5.15: We will use the shorthand $\mathfrak{r}_{j'\ldots j}(p) := \mathfrak{r}_{j'} \circ \ldots \circ \mathfrak{r}_j(p)$. To show that a reduction system R is reduction finite, we will use the following argument.

Let $(\mathfrak{r}_j)_{j\in\mathbb{N}}$ be an infinite sequence of basic reductions and let p be a path. A *branch of* $(\mathfrak{r}_{j\ldots 0}(p))_{j\in\mathbb{N}}$ will refer to a sequence of paths $(p_j)_{j\geq -1}$ such that $p_{-1} = p$ and for all $j \geq 0$, the path p_j appears with a non zero coefficient in the linear combination $\mathfrak{r}_j(p_{j-1})$.

Suppose that $(\mathfrak{r}_j)$ and p are such that for all integers $n_1 \in \mathbb{N}$, there exists $n \geq n_1$ such that $\mathfrak{r}_{n\ldots 0}(p) \neq \mathfrak{r}_{n_1\ldots 0}(p)$. We choose inductively a branch (p_j) for which for all $j_0 \geq -1$ and $j_1 > j_0$, there exists $j \geq j_1$ such that $r_{j\ldots(j_0+1)}(p_{j_0}) \neq r_{j_1\ldots(j_0+1)}(p_{j_0})$.

Suppose that $p_{-1}, \ldots, p_{j_0}$ are already chosen, and let $r_{j_0+1}(p_{j_0}) = \sum_{l=1}^{r} \lambda_l p^l_{j_0+1}$ for $\lambda_1, \ldots, \lambda_r \in K^*$. If $\forall l \in \{1, \ldots, r\}$, $\exists j^l_1 > j_0 + 1$, $\forall j \geq j^l_1$, $r_{j\ldots(j_0+2)}(p^l_{j_0+1}) = r_{j^l_1\ldots(j_0+2)}(p^l_{j_0+1})$ then for all $j \geq j_1$ where $j_1 = \max\{j^l_1 \mid 1 \leq l \leq r\}$,

$$\mathfrak{r}_{j\ldots(j_0+1)}(p_{j_0}) = \sum_{l=1}^{r} \lambda_l \mathfrak{r}_{j\ldots(j_0+2)}(p^l_{j_0+1}) = \sum_{l=1}^{r} \lambda_l \mathfrak{r}_{j_1\ldots(j_0+2)}(p^l_{j_0+1}) = \mathfrak{r}_{j_1\ldots(j_0+1)}(p_{j_0}),$$

a contradiction. Thus there exists l such that $\forall j_1 > j_0+1$, $\exists j \geq j_1$, $r_{j\ldots(j_0+2)}(p^l_{j_0+1}) \neq r_{j_1\ldots(j_0+2)}(p^l_{j_0+1})$, and we can choose $p_{j_0+1} = p^l_{j_0+1}$. By construction, (p_j) cannot be constant after some rank j_0, otherwise this would imply that $\mathfrak{r}_{j\ldots(j_0+1)}(p_{j_0}) = p_{j_0}$ for all $j \geq j_0 + 1$ since (p_j) is a branch.

This construction tells us that, if every branch must eventually be constant, then R must be reduction finite.

In order to show the proposition, we need to find a basis for the pinched gentle algebra $KQ/\langle I\rangle$ given by $Q_0 = \{1, \ldots, n\}, Q_1^g = \emptyset$ and with unique ordered set of vertices $C = Q_0$.

The case $n = 1$ is trivial. We first show the case $n = 2$. The set

$$R = \{(s_1, \varphi_1) = (\gamma_{kk}\alpha_i, {}_k\alpha_i\gamma_i), (s_2, \varphi_2) = ({}_k\alpha_{ii}\alpha_k, \gamma_k^2 - e_k) \mid i \neq k \in \{1, 2\}\}$$

is a reduction system for KQ whose associated two-sided ideal is I and whose set of irreducible paths coincides with the sets given in the proposition. Let's verify that it is reduction finite.

Let p be a path, $(\mathfrak{r}_j)_{j\in\mathbb{N}}$ be an infinite sequence of basic reductions, and let $(p_j)_j$ be a branch of $(\mathfrak{r}_{j\ldots 0}(p))_{j\in\mathbb{N}}$. When applying a basic reduction $r_{j+1}(p_j) = \sum_{l=1}^{r} \lambda_l p_{j+1}^l$, the number of arrows of the form ${}_{h+1}\alpha_h$ in each p_{j+1}^l must be less or equal than the number of arrows of this form in p_j. Thus for the elements of (p_j) this number must eventually be constant, meaning that there exists $j_0 \in \mathbb{N}$ such that for every $\mathfrak{r}_j$ of type s_2 with $j > j_0$, $p_j = \mathfrak{r}_j(p_{j-1}) = p_{j-1}$. But then since there cannot be an infinite sequence of terms for which an arrow γ_h goes to the right, all reductions of type s_1 must eventually act as the identity. Hence, the branch is eventually constant, and R must be reduction finite.

There are two overlap ambiguities given by $\gamma_{kk}\alpha_{ii}\alpha_k$ and ${}_i\alpha_{kk}\alpha_{ii}\alpha_k$ for $i \neq k$, and they are resolved in the following way:

$$\begin{aligned}
&\gamma_{kk}\alpha_{ii}\alpha_k \xmapsto{s_2} \gamma_k(\gamma_k^2 - e_k)\\
&\gamma_{kk}\alpha_{ii}\alpha_k \xmapsto{s_1} {}_k\alpha_i\gamma_{ii}\alpha_k \xmapsto{s_1} {}_k\alpha_{ii}\alpha_k\gamma_k \xmapsto{s_2} (\gamma_k^2 - e_k)\gamma_k
\end{aligned}$$

$$\begin{aligned}
&{}_i\alpha_{kk}\alpha_{ii}\alpha_k \xmapsto{s_2} {}_i\alpha_k(\gamma_k^2 - e_k)\\
&{}_i\alpha_{kk}\alpha_{ii}\alpha_k \xmapsto{s_2} (\gamma_i^2 - e_i)_i\alpha_k \xmapsto{s_1} {}_i\alpha_k(\gamma_k^2 - e_k)
\end{aligned}$$

Now suppose $n \geq 3$. Let R be the set containing the following couples:

- for each couple $i \neq k$ consecutive in $\{1, \ldots, n\}$,

$$\begin{aligned}
(s_1, \varphi_{s_1}) &= (\gamma_{kk}\alpha_i, {}_k\alpha_i\gamma_i),\\
(s_2, \varphi_{s_2}) &= ({}_i\alpha_{kk}\alpha_i, \gamma_i^2 - e_i),
\end{aligned}$$

- for each couple $i \neq k \in \{1, \ldots, n\}$,

$$(s_3, \varphi_{s_3}) = ({}_k\alpha_i^- \gamma_i^{l^+}, {}_k\alpha_i^+(\gamma_i - e_i)^{l^-} - {}_k\alpha_i^- \sum_{s=0}^{l^+-1} \binom{l^+}{s} \gamma_i^s),$$

- and for each $i \in \{1, \ldots, n\}$,

$$(s_4, \varphi_{s_4}) = ({}_i\alpha_i^+, (\gamma_i - e_i)(\gamma_i + e_i)^{n-1}).$$

One can check that R is a reduction system for KQ. Moreover, the two-sided ideal associated to R is I, and the set of irreducible paths coincides with the sets given in the proposition. The following argument shows that it is reduction finite.

Let p be a path, $(\mathfrak{r}_j)_{j\in\mathbb{N}}$ be an infinite sequence of basic reductions, and $(p_j)_j$ be a branch of $(\mathfrak{r}_{j\dots 0}(p))_{j\in\mathbb{N}}$. The length $l(p_j)$ is the number of arrows that p_j contains. Let $l_\alpha(p_j)$ be the number of arrows of type ${}_k\alpha_i$ that p_j contains, and $l_{\alpha^-}(p_j)$ be the number of counter-clockwise arrows ${}_{i-1}\alpha_i$ that it contains. The reductions are such that for all $j \geq -1$, $l_{\alpha^-}(p_{j+1}) \leq l_{\alpha^-}(p_j)$.

Suppose that there is infinitely many reduction of type s_3 in $(\mathfrak{r}_j)$. Then there cannot be an infinite number of reductions of type s_3 such that p_j is obtained from p_{j-1} by replacing ${}_k\alpha_i^-\gamma_i^{l^+}$ by a term of ${}_k\alpha_i^+(\gamma_i - e_i)^{l^-}$, since l_{α^-} is bounded below by zero. Once this replacement does not occur anymore, the length $l(p_{j+1})$ of each term p_{j+1} can only be less or equal than the length of p_j. From this we deduce that ${}_k\alpha_i^-\gamma_i^{l^+}$ cannot be replaced an infinite number of times by a term of ${}_k\alpha_i^-\sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma_i^s$, since it would make the length l strictly decrease an infinite number of times. Thus all reductions of type s_3 must eventually act as the identity.

After this, since reductions of type s_2 and s_4 make the quantity l_α strictly decrease, they too must eventually act as the identity. Finally, there can only be a finite number of transformations for which an arrow γ_h goes to the right, so reductions of type s_1 eventually act trivially. This shows that the branch is eventually constant, and we deduce that R is reduction finite.

It remains to show that all overlap ambiguities of R are resolvable. For a and b in $\{1,\dots,n\}$, we introduce the notation $[a \triangleright b]$ for the set $\{a, a+1, \dots, b-1, b\}$, where the integers are taken modulo n. In particular $[a \triangleright a] = \{a\}$ and $[a+1 \triangleright a] = \{1,\dots,n\}$ for all a. The following is a complete list of overlaps for R, where $i \neq k$.

$(s_1, s_2) : \gamma_{kk}\alpha_{ii}\alpha_k$

$(s_1, s_3) : \gamma_{kk}\alpha_{k+1}\dots{}_{i-1}\alpha_i\gamma_i^{l^+}$

$(s_1, s_4) : \gamma_{kk}\alpha_{k-1}\dots{}_{k+1}\alpha_k$

$(s_2, s_2) : {}_i\alpha_{kk}\alpha_{ii}\alpha_k$

$(s_2, s_3) : {}_{k+1}\alpha_{kk}\alpha_{k+1}\dots{}_{i-1}\alpha_i\gamma_i^{l^+}$

$(s_2, s_4) : {}_{k-1}\alpha_{kk}\alpha_{k-1}\dots{}_{k+1}\alpha_k$

$(s_3, s_1) : {}_k\alpha_i^-\gamma_i^{l^+-1}\gamma_{ii}\alpha_{i\pm1}$ with $l^+ \geq 1$

$(s_3, s_2) : {}_{i+1}\alpha_{i-1}^-{}_{i-1}\alpha_{ii}\alpha_{i-1}$

$(s_3, s_3) : {}_k\alpha_h^-{}_h\alpha_i^-\gamma_i^{l^+}\gamma_i^{\eta^+-l^+}$ for $i \neq k+1$, and with $h \in [k+1 \triangleright i-1]$

$(s_4, s_2) : {}_i\alpha_{i-1}\dots{}_{i+1}\alpha_{ii}\alpha_{i+1}$

$(s_4, s_4) : {}_i\alpha_h^+{}_h\alpha_i^+{}_i\alpha_h^+$ with $h \in [i+1 \triangleright i-1]$

We now resolve each of them.

$\triangleright$ (s_1, s_2) : First

$$\gamma_{kk}\alpha_{ii}\alpha_k \xmapsto{s_2} \gamma_k(\gamma_k^2 - e_k)$$

Then

$$\gamma_{kk}\alpha_{ii}\alpha_k \xmapsto{s_1} {}_k\alpha_i\gamma_{ii}\alpha_k \xmapsto{s_1} {}_k\alpha_{ii}\alpha_k\gamma_k \xmapsto{s_2} (\gamma_k^2 - e_k)\gamma_k$$

$\triangleright$ (s_1, s_3) : First

$$\gamma_{kk}\alpha_{k+1}\dots{}_{i-1}\alpha_i\gamma_i^{l^+} \xmapsto{s_1} {}_k\alpha_i^-\gamma_i^{l^++1} \xmapsto{s_3} {}_k\alpha_i^+(\gamma_i - e_i)^{l^-}\gamma_i - {}_k\alpha_i^-\sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma_i^{s+1}$$

Then

$$\gamma_{kk}\alpha_{k+1}\cdots{}_{i-1}\alpha_i\gamma_i^{l^+} \xmapsto{s_3} \gamma_{kk}\alpha_i^+(\gamma_i - e_i)^{l^-} - \gamma_{kk}\alpha_i^- \sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma_i^s$$
$$\xmapsto{s_1} {}_k\alpha_i^+(\gamma_i - e_i)^{l^-}\gamma_i - {}_k\alpha_i^- \sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma_i^{s+1}$$

▷ (s_1, s_4) : First

$$\gamma_{kk}\alpha_{k-1}\cdots{}_{k+1}\alpha_k \xmapsto{s_4} \gamma_k(\gamma_k - e_k)(\gamma_k + e_k)^{n-1}$$

Then

$$\gamma_{kk}\alpha_{k-1}\cdots{}_{k+1}\alpha_k \xmapsto{s_1} {}_k\alpha_{k-1}\gamma_{k-1k-1}\alpha_k^+ \xmapsto{s_1} {}_k\alpha_k^+\gamma_k \xmapsto{s_4} \gamma_k(\gamma_k - e_k)(\gamma_k + e_k)^{n-1}$$

▷ (s_2, s_2) : First

$${}_i\alpha_{kk}\alpha_{ii}\alpha_k \xmapsto{s_2} {}_i\alpha_k(\gamma_k^2 - e_k)$$

Then

$${}_i\alpha_{kk}\alpha_{ii}\alpha_k \xmapsto{s_2} (\gamma_i^2 - e_i)_i\alpha_k \xmapsto{s_1} {}_i\alpha_k(\gamma_k^2 - e_k)$$

▷ (s_2, s_3) : We will resolve ${}_{k+1}\alpha_{kk}\alpha_{k+1}\cdots{}_{i-1}\alpha_i\gamma_i^{l^+}$. First suppose that $i = k+1$. We have

$$({}_{k+1}\alpha_{kk}\alpha_{k+1})\gamma_{k+1}^{n-2} \xmapsto{s_2} (\gamma_{k+1}^2 - e_{k+1})\gamma_{k+1}^{n-2}$$

and

$${}_{k+1}\alpha_k({}_k\alpha_{k+1}\gamma_{k+1}^{n-2}) \xmapsto{s_3} {}_{k+1}\alpha_{k+1}^+ - ({}_{k+1}\alpha_{kk}\alpha_{k+1})\sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{k+1}^s$$
$$\xmapsto{s_2} {}_{k+1}\alpha_{k+1}^+ - (\gamma_{k+1}^2 - e_{k+1})\sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{k+1}^s$$
$$\xmapsto{s_4} (\gamma_{k+1} - e_{k+1})(\gamma_{k+1} + e_{k+1})^{n-1} - (\gamma_{k+1}^2 - e_{k+1})\sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{k+1}^s$$
$$= (\gamma_{k+1}^2 - e_{k+1})\Big((\gamma_{k+1} + e_{k+1})^{n-2} - \sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{k+1}^s\Big)$$
$$= (\gamma_{k+1}^2 - e_{k+1})\gamma_{k+1}^{n-2}$$

Now suppose that $i \neq k+1$. First we have

$$({}_{k+1}\alpha_{kk}\alpha_{k+1})\cdots{}_{i-1}\alpha_i\gamma_i^{l^+} \xmapsto{s_1 s_2} ({}_{k+1}\alpha_i^-\gamma_i^{l^+ +1})\gamma_i - {}_{k+1}\alpha_i^-\gamma_i^{l^+}$$
$$\xmapsto{s_3} {}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^- -1}\gamma_i - {}_{k+1}\alpha_i^- \sum_{s=0}^{l^+}\binom{l^+ + 1}{s}\gamma_i^{s+1} - {}_{k+1}\alpha_i^-\gamma_i^{l^+}$$
$$= (*) - (l^+ + 1)_{k+1}\alpha_i^-\gamma_i^{l^+ +1}$$

where $(*) = {}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^- -1}\gamma_i - {}_{k+1}\alpha_i^- \sum_{s=0}^{l^+-1} \binom{l^++1}{s}\gamma_i^{s+1} - {}_{k+1}\alpha_i^- \gamma_i^{l^+}$. Applying s_3 gives:

$$\begin{aligned}(*) - (l^+ + 1)_{k+1}\alpha_i^-\gamma_i^{l^++1} &\xmapsto{s_3} (*) - (l^+ + 1)_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^- -1} + (l^+ + 1)_{k+1}\alpha_i^- \sum_{s=0}^{l^+} \binom{l^+ + 1}{s}\gamma_i^s \\ &= {}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^- -1}(\gamma_i - (l^+ + 1)e_i) + {}_{k+1}\alpha_i^-[1]\end{aligned}$$

where $[1] = (l^+ + 1) \sum_{s=0}^{l^+} \binom{l^++1}{s}\gamma_i^s - \sum_{s=0}^{l^+-1} \binom{l^++1}{s}\gamma_i^{s+1} - \gamma_i^{l^+}$.

Then we have

$$\begin{aligned}{}_{k+1}\alpha_k({}_k\alpha_{k+1} \dots {}_{i-1}\alpha_i\gamma_i^{l^+}) &\xmapsto{s_3} {}_{k+1}\alpha_{k k}\alpha_i^+(\gamma_i - e_i)^{l^-} - {}_{k+1}\alpha_{k k}\alpha_i^- \sum_{s=0}^{l^+-1} \binom{l^+}{s}\gamma_i^s \\ &= {}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^+} - ({}_{k+1}\alpha_{k k}\alpha_{k+1})_{k+1}\alpha_i^- \sum_{s=0}^{l^+-1} \binom{l^+}{s}\gamma_i^s \\ &\xmapsto{s_2} {}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^+} - {}_{k+1}\alpha_i^- \sum_{s=0}^{l^+-1} \binom{l^+}{s}\gamma_i^{s+2} + {}_{k+1}\alpha_i^- \sum_{s=0}^{l^+-1} \binom{l^+}{s}\gamma_i^s \\ &= (**) - l^+{}_{k+1}\alpha_i^-\gamma_i^{l^++1}\end{aligned}$$

where $(**) = {}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^+} - {}_{k+1}\alpha_i^- \sum_{s=0}^{l^+-2} \binom{l^+}{s}\gamma_i^{s+2} + {}_{k+1}\alpha_i^- \sum_{s=0}^{l^+-1} \binom{l^+}{s}\gamma_i^s$. Applying s_3 gives:

$$\begin{aligned}(**) - l^+{}_{k+1}\alpha_i^-\gamma_i^{l^++1} &\xmapsto{s_3} (**) - l^+{}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^- -1} + l^+{}_{k+1}\alpha_i^- \sum_{s=0}^{l^+} \binom{l^+ + 1}{s}\gamma_i^s \\ &= {}_{k+1}\alpha_i^+(\gamma_i - e_i)^{l^+ -1}(\gamma_i - (l^+ + 1)e_i) + {}_{k+1}\alpha_i^-[2]\end{aligned}$$

where $[2] = l^+ \sum_{s=0}^{l^+} \binom{l^++1}{s}\gamma_i^s - \sum_{s=0}^{l^+-2} \binom{l^+}{s}\gamma_i^{s+2} + \sum_{s=0}^{l^+-1} \binom{l^+}{s}\gamma_i^s$.

Let us show the egality between $[1]$ and $[2]$. On one hand we have

$$\begin{aligned}[1] &= l^+ \sum_{s=0}^{l^+} \binom{l^+ + 1}{s}\gamma_i^s + \sum_{s=0}^{l^+} \binom{l^+ + 1}{s}\gamma_i^s - \sum_{s=1}^{l^+} \binom{l^+ + 1}{s-1}\gamma_i^s - \gamma_i^{l^+} \\ &= l^+ \sum_{s=0}^{l^+} \binom{l^+ + 1}{s}\gamma_i^s + [\binom{l^+ + 1}{l^+} - \binom{l^+ + 1}{l^+ - 1}]\gamma_i^{l^+} + \sum_{s=1}^{l^+-1} [\binom{l^+ + 1}{s} - \binom{l^+ + 1}{s-1}]\gamma_i^s + e_i - \gamma_i^{l^+} \\ &= l^+ \sum_{s=0}^{l^+} \binom{l^+ + 1}{s}\gamma_i^s + [l^+ - \binom{l^+ + 1}{l^+ - 1}]\gamma_i^{l^+} + \sum_{s=2}^{l^+-1} [\binom{l^+ + 1}{s} - \binom{l^+ + 1}{s-1}]\gamma_i^s + l^+\gamma_i + e_i\end{aligned} \tag{3.19}$$

On the other hand,

$$\begin{aligned}[2] &= l^+ \sum_{s=0}^{l^+} \binom{l^+ + 1}{s} \gamma_i^s - \sum_{s=2}^{l^+} \binom{l^+}{s-2} \gamma_i^s + \sum_{s=0}^{l^+-1} \binom{l^+}{s} \gamma_i^s \\ &= l^+ \sum_{s=0}^{l^+} \binom{l^+ + 1}{s} \gamma_i^s - \binom{l^+}{l^+ - 2} \gamma_i^{l^+} + \sum_{s=2}^{l^+-1} [\binom{l^+}{s} - \binom{l^+}{s-2}] \gamma_i^s + l^+ \gamma_i + e_i \end{aligned} \tag{3.20}$$

Finally using $\binom{b-1}{a-1} + \binom{b-1}{a} = \binom{b}{a}$, we have $l^+ - \binom{l^+ +1}{l^+ -1} = \binom{l^+}{l^+ -1} - \binom{l^+ +1}{l^+ -1} = -\binom{l^+}{l^+ -2}$ and

$$\binom{l^+}{s} - \binom{l^+}{s-2} = \binom{l^+ + 1}{s} - \binom{l^+}{s-1} - \binom{l^+}{s-2} = \binom{l^+ + 1}{s} - \binom{l^+ + 1}{s-1}$$

▷ (s_2, s_4) : First

$${}_{k-1}\alpha_{kk}\alpha_{k-1} \dots {}_{k+1}\alpha_k \xmapsto{s_2} (\gamma_{k-1}^2 - e_{k-1})_{k-1}\alpha_k^+ \xmapsto{s_1} {}_{k-1}\alpha_k^+ (\gamma_k^2 - e_k)$$

Then

$$\begin{aligned} {}_{k-1}\alpha_{kk}\alpha_{k-1} \dots {}_{k+1}\alpha_k &\xmapsto{s_4} {}_{k-1}\alpha_k (\gamma_k - e_k)(\gamma_k + e_k)^{n-1} = {}_{k-1}\alpha_k (\gamma_k + e_k)^{n-2} (\gamma_k^2 - e_k) \\ &= ({}_{k-1}\alpha_k \gamma_k^{n-2} + {}_{k-1}\alpha_k \sum_{s=0}^{n-3} \binom{n-2}{s} \gamma_k^s)(\gamma_k^2 - e_k) \\ &\xmapsto{s_3} (({}_{k-1}\alpha_k^+ - {}_{k-1}\alpha_k \sum_{s=0}^{n-3} \binom{n-2}{s} \gamma_k^s) + {}_{k-1}\alpha_k \sum_{s=0}^{n-3} \binom{n-2}{s} \gamma_k^s)(\gamma_k^2 - e_k) \\ &= {}_{k-1}\alpha_k^+ (\gamma_k^2 - e_k) \end{aligned}$$

▷ (s_3, s_1) : We will resolve ${}_k\alpha_i^- \gamma_i^{l^+ -1} \gamma_{ii}\alpha_h$ with $l^+ > 0$, for $h = i+1$ and then for $h = i-1$. First we have

$$\begin{aligned} ({}_k\alpha_i^- \gamma_i^{l^+ -1} \gamma_i)_i \alpha_{i+1} &\xmapsto{s_1 s_3} {}_k\alpha_i^+ {}_i\alpha_{i+1} (\gamma_{i+1} - e_{i+1})^{l^-} - {}_k\alpha_i^- {}_i\alpha_{i+1} \sum_{s=0}^{l^+ -1} \binom{l^+}{s} \gamma_{i+1}^s \\ &= {}_k\alpha_{i+1}^+ ({}_{i+1}\alpha_{ii}\alpha_{i+1})(\gamma_{i+1} - e_{i+1})^{l^-} - {}_k\alpha_{i+1}^- \sum_{s=0}^{l^+ -1} \binom{l^+}{s} \gamma_{i+1}^s \\ &\xmapsto{s_2} {}_k\alpha_{i+1}^+ (\gamma_{i+1} - e_{i+1})^{l^- +1} (\gamma_{i+1} + e_{i+1}) - {}_k\alpha_{i+1}^- \sum_{s=0}^{l^+ -1} \binom{l^+}{s} \gamma_{i+1}^s \end{aligned}$$

Then

$$\begin{aligned} {}_k\alpha_i^- \gamma_i^{l^+ -1} (\gamma_{ii}\alpha_{i+1}) &\xmapsto{s_1} {}_k\alpha_i^- \gamma_i^{l^+ -1} {}_i\alpha_{i+1} \gamma_{i+1} \xmapsto{s_1} {}_k\alpha_i^- {}_i\alpha_{i+1} \gamma_{i+1}^{l^+} = ({}_k\alpha_{i+1}^- \gamma_{i+1}^{l^+ -1}) \gamma_{i+1} \\ &\xmapsto{s_3} {}_k\alpha_{i+1}^+ (\gamma_{i+1} - e_{i+1})^{l^- +1} \gamma_{i+1} - {}_k\alpha_{i+1}^- \sum_{s=0}^{l^+ -2} \binom{l^+ - 1}{s} \gamma_{i+1}^{s+1} \\ &= (*) + {}_k\alpha_{i+1}^- \gamma_{i+1}^{l^+ -1} \end{aligned}$$

where $(*) = {}_k\alpha^+_{i+1}(\gamma_{i+1} - e_{i+1})^{l^-+1}\gamma_{i+1} - l^+{}_k\alpha^-_{i+1}\gamma_i^{l^+-1} - {}_k\alpha^-_{i+1}\sum_{s=0}^{l^+-3}\binom{l^+-1}{s}\gamma_{i+1}^{s+1}$. Applying s_3 gives:

$$
\begin{aligned}
(*) + {}_k\alpha^-_{i+1}\gamma_{i+1}^{l^+-1} &\xrightarrow{s_3} (*) + {}_k\alpha^+_{i+1}(\gamma_{i+1} - e_{i+1})^{l^-+1} - {}_k\alpha^-_{i+1}\sum_{s=0}^{l^+-2}\binom{l^+-1}{s}\gamma_{i+1}^{s} \\
&= \big({}_k\alpha^+_{i+1}(\gamma_{i+1} - e_{i+1})^{l^-+1}\gamma_{i+1} + {}_k\alpha^+_{i+1}(\gamma_{i+1} - e_{i+1})^{l^-+1}\big) - l^+{}_k\alpha^-_{i+1}\gamma_i^{l^+-1} \\
&\qquad - {}_k\alpha^-_{i+1}\sum_{s=0}^{l^+-3}\binom{l^+-1}{s}\gamma_{i+1}^{s+1} - {}_k\alpha^-_{i+1}\sum_{s=0}^{l^+-2}\binom{l^+-1}{s}\gamma_{i+1}^{s} \\
&= {}_k\alpha^+_{i+1}(\gamma_{i+1} - e_{i+1})^{l^-+1}(\gamma_{i+1} + e_{i+1}) - l^+{}_k\alpha^-_{i+1}\gamma_i^{l^+-1} - {}_k\alpha^-_{i+1} \\
&\qquad - {}_k\alpha^-_{i+1}\sum_{s=1}^{l^+-2}\binom{l^+-1}{s-1}\gamma_{i+1}^{s} - {}_k\alpha^-_{i+1}\sum_{s=1}^{l^+-2}\binom{l^+-1}{s}\gamma_{i+1}^{s} \\
&= {}_k\alpha^+_{i+1}(\gamma_{i+1} - e_{i+1})^{l^-+1}(\gamma_{i+1} + e_{i+1}) - l^+{}_k\alpha^-_{i+1}\gamma_i^{l^+-1} - {}_k\alpha^-_{i+1}\sum_{s=1}^{l^+-2}\binom{l^+}{s}\gamma_{i+1}^{s} - {}_k\alpha^-_{i+1} \\
&= {}_k\alpha^+_{i+1}(\gamma_{i+1} - e_{i+1})^{l^-+1}(\gamma_{i+1} + e_{i+1}) - {}_k\alpha^-_{i+1}\sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma_{i+1}^{s}
\end{aligned}
$$

using $\binom{l^+-1}{s-1} + \binom{l^+-1}{s} = \binom{l^+}{s}$.

Now lets do the case $h = i-1$. First suppose that $k = i-1$. We have

$$
{}_{i-1}\alpha_i\gamma_i^{n-3}(\gamma_{ii}\alpha_{i-1}) \xrightarrow{s_1} ({}_{i-1}\alpha_{ii}\alpha_{i-1})\gamma_{i-1}^{n-2} \xrightarrow{s_2} (\gamma_{i-1}^2 - e_{i-1})\gamma_{i-1}^{n-2}
$$

and

$$
\begin{aligned}
({}_{i-1}\alpha_i\gamma_i^{n-3}\gamma_i)_i\alpha_{i-1} &\xrightarrow{s_1 s_3} {}_{i-1}\alpha_i^+{}_i\alpha_{i-1} - ({}_{i-1}\alpha_{ii}\alpha_{i-1})\sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{i-1}^s \\
&\xrightarrow{s_2} {}_{i-1}\alpha^+_{i-1} - (\gamma_{i-1}^2 - e_{i-1})\sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{i-1}^s \\
&\xrightarrow{s_4} (\gamma_{i-1} - e_{i-1})(\gamma_{i-1} + e_{i-1})^{n-1} - (\gamma_{i-1}^2 - e_{i-1})\sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{i-1}^s \\
&= (\gamma_{i-1}^2 - e_{i-1})\Big((\gamma_{i-1} + e_{i-1})^{n-2} - \sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{i-1}^s\Big) = (\gamma_{i-1}^2 - e_{i-1})\gamma_{i-1}^{n-2}
\end{aligned}
$$

Now suppose $k \neq i-1$. First

$$
\begin{aligned}
({}_k\alpha_i^-\gamma_i^{l^+-1}\gamma_i)_i\alpha_{i-1} &\xrightarrow{s_1 s_3} {}_k\alpha^+_{i-1}(\gamma_{i-1} - e_{i-1})^{l^-} - {}_k\,\alpha^-_{i-1}({}_{i-1}\alpha_{ii}\alpha_{i-1})\sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma_{i-1}^s \\
&\xrightarrow{s_2} {}_k\alpha^+_{i-1}(\gamma_{i-1} - e_{i-1})^{l^-} - {}_k\,\alpha^-_{i-1}(\gamma_{i-1}^2 - e_{i-1})\sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma_{i-1}^s \\
&= (*) - l^+{}_k\,\alpha^-_{i-1}\gamma_{i-1}^{l^++1}
\end{aligned}
$$

where $(*) = {}_k\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{l^-} - {}_k\alpha^-_{i-1}\sum_{s=0}^{l^+-2}\binom{l^+}{s}\gamma^{s+2}_{i-1} + {}_k\alpha^-_{i-1}\sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma^{s}_{i-1}$. Applying s_3 gives:

$$\begin{aligned}
(*) - l^+{}_k\,\alpha^-_{i-1}\gamma^{l^++1}_{i-1} &\xmapsto{s_3} (*) - l^+{}_k\,\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{l^--1} + l^+{}_k\alpha^-_{i-1}\sum_{s=0}^{l^+}\binom{l^++1}{s}\gamma^s_{i-1}\\
&= {}_k\,\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{l^--1}(\gamma_{i-1}-(l^++1)e_{i-1})\\
&\quad + {}_k\alpha^-_{i-1}[l^+\sum_{s=0}^{l^+}\binom{l^++1}{s}\gamma^s_{i-1} - \sum_{s=0}^{l^+-2}\binom{l^+}{s}\gamma^{s+2}_{i-1} + \sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma^s_{i-1}]
\end{aligned}$$

Then

$$\begin{aligned}
{}_k\alpha^-_i\gamma_i^{l^+-1}(\gamma_{ii}\alpha_{i-1}) &\xmapsto{s_1} {}_k\alpha^-_{i-1}({}_{i-1}\alpha_{ii}\alpha_{i-1})\gamma^{l^+}_{i-1} \xmapsto{s_2} ({}_k\alpha^-_{i-1}\gamma^{l^++1}_{i-1})\gamma_{i-1} - {}_k\alpha^-_{i-1}\gamma^{l^+}_{i-1}\\
&\xmapsto{s_3} {}_k\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{l^--1}\gamma_{i-1} - {}_k\alpha^-_{i-1}\sum_{s=0}^{l^+}\binom{l^++1}{s}\gamma^{s+1}_{i-1} - {}_k\alpha^-_{i-1}\gamma^{l^+}_{i-1}\\
&= (**) - (l^++1){}_k\alpha^-_{i-1}\gamma^{l^++1}_{i-1}
\end{aligned}$$

where $(**) = {}_k\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{l^--1}\gamma_{i-1} - {}_k\alpha^-_{i-1}\sum_{s=0}^{l^+-1}\binom{l^++1}{s}\gamma^{s+1}_{i-1} - {}_k\alpha^-_{i-1}\gamma^{l^+}_{i-1}$. Applying s_3 gives:

$$\begin{aligned}
(**) - (l^++1){}_k\alpha^-_{i-1}\gamma^{l^++1}_{i-1} &\xmapsto{s_3} (**) - (l^++1){}_k\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{l^--1} + (l^++1){}_k\alpha^-_{i-1}\sum_{s=0}^{l^+}\binom{l^++1}{s}\gamma^s_{i-1}\\
&= {}_k\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{l^--1}(\gamma_{i-1}-(l^++1)e_{i-1})\\
&\quad + {}_k\alpha^-_{i-1}[(l^++1)\sum_{s=0}^{l^+}\binom{l^++1}{s}\gamma^s_{i-1} - \sum_{s=0}^{l^+-1}\binom{l^++1}{s}\gamma^{s+1}_{i-1} - \gamma^{l^+}_{i-1}]
\end{aligned}$$

This is similar to Equation 3.19 and 3.20.

▷ (s_3, s_2) : First

$$({}_{i+1}\alpha^-_{i-1}{}_{i-1}\alpha_i){}_i\alpha_{i-1} \xmapsto{s_3} {}_{i+1}\alpha_i(\gamma_i-e_i)^{n-2}{}_i\alpha_{i-1} \xmapsto{s_1} {}_{i+1}\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{n-2}$$

Then

$$\begin{aligned}
{}_{i+1}\alpha^-_{i-1}({}_{i-1}\alpha_{ii}\alpha_{i-1}) &\xmapsto{s_2} ({}_{i+1}\alpha^-_{i-1}\gamma_{i-1})\gamma_{i-1} - {}_{i+1}\alpha^-_{i-1}\\
&\xmapsto{s_3} {}_{i+1}\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{n-3}\gamma_{i-1} - ({}_{i+1}\alpha^-_{i-1}\gamma_{i-1}) - {}_{i+1}\alpha^-_{i-1}\\
&\xmapsto{s_3} {}_{i+1}\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{n-3}\gamma_{i-1} - ({}_{i+1}\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{n-3} - {}_{i+1}\alpha^-_{i-1}) - {}_{i+1}\alpha^-_{i-1}\\
&= {}_{i+1}\alpha^+_{i-1}(\gamma_{i-1}-e_{i-1})^{n-3}(\gamma_{i-1}-e_{i-1})
\end{aligned}$$

▷ (s_3, s_3) : Suppose $i \neq k+1$ and let $h \in [k+1 \triangleright i-1]$. First

$$({}_k\alpha^-_h{}_h\alpha^-_i\gamma^{l^+}_i)\gamma_i^{\eta^+-l^+} \xmapsto{s_3} ({}_k\alpha^+_i(\gamma_i-e_i)^{l^-} - {}_k\alpha^-_i\sum_{s=0}^{l^+-1}\binom{l^+}{s}\gamma^s_i)\gamma_i^{\eta^+-l^+} =: (a)$$

Then

$$ {}_k\alpha_h^-\big({}_h\alpha_i^-\gamma_i^{l^+}\gamma_i^{\eta^+-l^+}\big) \xmapsto{s_3} {}_k\alpha_h^-\big({}_h\alpha_i^+(\gamma_i-e_i)^{\eta^-} - {}_h\alpha_i^-\sum_{s=0}^{\eta^+-1}\binom{\eta^+}{s}\gamma_i^s\big) =: (b) $$

We will construct a sequence of reductions $\mathfrak{r}$ from (b) to (a). Let $j\in[k\triangleright h-1]$, δ^+ be the length of ${}_j\alpha_{j-1}\dots{}_{i+1}\alpha_i$ minus one and $d=\eta^+-\delta^+-1$. First

$$ {}_k\alpha_j^-\big({}_j\alpha_{j+1\,j+1}\alpha_i^-\gamma_i^{\delta^+}\big)\gamma_i\gamma_i^d \xmapsto{s_3} {}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}\gamma_i^{d+1} - {}_k\alpha_i^-\sum_{s=0}^{\delta^+-1}\binom{\delta^+}{s}\gamma_i^s\gamma_i^{d+1} $$

Then

$$ \begin{aligned} {}_k\alpha_j^-\,{}_j\alpha_{j+1}\big({}_{j+1}\alpha_i^-\gamma_i^{\delta^+}\gamma_i\big)\gamma_i^d &\xmapsto{s_3} {}_k\alpha_j^-\,{}_j\alpha_{j+1}\big({}_{j+1}\alpha_i^+(\gamma_i-e_i)^{\delta^--1} - {}_{j+1}\alpha_i^-\sum_{s=0}^{\delta^+}\binom{\delta^++1}{s}\gamma_i^s\big)\gamma_i^d \\ &= {}_k\alpha_j^-\big({}_j\alpha_{j+1\,j+1}\alpha_j\big){}_j\alpha_i^+(\gamma_i-e_i)^{\delta^--1}\gamma_i^d - {}_k\alpha_i^-\sum_{s=0}^{\delta^+}\binom{\delta^++1}{s}\gamma_i^{s+d} \\ &\xmapsto{s_1s_2} {}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}(\gamma_i+e_i)\gamma_i^d - (\delta^++1){}_k\alpha_i^-\gamma_i^{\delta^++d} - {}_k\alpha_i^-\sum_{s=0}^{\delta^+-1}\binom{\delta^++1}{s}\gamma_i^{s+d} \\ &= (*) - {}_k\alpha_j^-\,{}_j\alpha_i^-\gamma_i^{\delta^+}\gamma_i^d \end{aligned} \tag{3.21} $$

where $(*)={}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}(\gamma_i+e_i)\gamma_i^d-\delta^+{}_k\alpha_i^-\gamma_i^{\delta^++d}-{}_k\alpha_i^-\sum_{s=0}^{\delta^+-1}\binom{\delta^++1}{s}\gamma_i^{s+d}$. Applying s_3 gives:

$$ \begin{aligned} (*) - {}_k\alpha_j^-\big({}_j\alpha_i^-\gamma_i^{\delta^+}\big)\gamma_i^d &\xmapsto{s_3} (A) - {}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}\gamma_i^d + {}_k\alpha_i^-\sum_{s=0}^{\delta^+-1}\binom{\delta^+}{s}\gamma_i^{s+d} \\ &= \big({}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}(\gamma_i+e_i)\gamma_i^d - {}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}\gamma_i^d\big) \\ &\qquad - \delta^+{}_k\alpha_i^-\gamma_i^{\delta^++d} - {}_k\alpha_i^-\sum_{s=0}^{\delta^+-1}\Big[\binom{\delta^++1}{s}-\binom{\delta^+}{s}\Big]\gamma_i^{s+d} \\ &= {}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}\gamma_i^{d+1} - \delta^+{}_k\alpha_i^-\gamma_i^{\delta^++d} - {}_k\alpha_i^-\sum_{s=1}^{\delta^+-1}\binom{\delta^+}{s-1}\gamma_i^{s+d} \\ &= {}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}\gamma_i^{d+1} - \delta^+{}_k\alpha_i^-\gamma_i^{\delta^++d} - {}_k\alpha_i^-\sum_{s=0}^{\delta^+-2}\binom{\delta^+}{s}\gamma_i^{s+1+d} \\ &= {}_k\alpha_j^-\,{}_j\alpha_i^+(\gamma_i-e_i)^{\delta^-}\gamma_i^{d+1} - {}_k\alpha_i^-\sum_{s=0}^{\delta^+-1}\binom{\delta^+}{s}\gamma_i^{s+d+1} \end{aligned} \tag{3.22} $$

using $\binom{\delta^+}{s-1}+\binom{\delta^+}{s}=\binom{\delta^++1}{s}$. Note that ${}_k\alpha_l^-$ should be replaced by e_k when $l=k$. Take $\mathfrak{r}$ to be the composition of each $\mathfrak{r}_l$ given by Equation 3.21 and 3.22, for $l\in[k+1\triangleright h]$.

$\triangleright$ (s_4,s_2) : First

$$ {}_i\alpha_{i-1}\dots{}_{i+1}\alpha_{ii}\alpha_{i+1} \xmapsto{s_2} {}_i\alpha_{i+1}^+(\gamma_{i+1}^2-e_{i+1}) $$

Then

$$\begin{aligned}
{}_i\alpha_{i-1}\cdots{}_{i+1}\alpha_{i}{}_i\alpha_{i+1} &\xmapsto{s_4} (\gamma_i - e_i)(\gamma_i + e_i)^{n-1}{}_i\alpha_{i+1} \\
&\xmapsto{s_1} {}_i\alpha_{i+1}(\gamma_{i+1} + e_{i+1})^{n-2}(\gamma_{i+1}^2 - e_{i+1}) \\
&= ({}_i\alpha_{i+1}\gamma_{i+1}^{n-2} + {}_i\alpha_{i+1}\sum_{s=0}^{n-3}\binom{n-2}{s}\gamma_{i+1}^s)(\gamma_{i+1}^2 - e_{i+1}) \\
&\xmapsto{s_3} {}_i\alpha_{i+1}^+(\gamma_{i+1}^2 - e_{i+1})
\end{aligned}$$

▷ (s_4, s_4) : Let $h \in [i+1 \triangleright i-1]$. First

$${}_i\alpha_h^+{}_h\alpha_i^+{}_i\alpha_h^+ \xmapsto{s_4} {}_i\alpha_h^+(\gamma_h - e_h)(\gamma_h + e_h)^{n-1}$$

Then

$${}_i\alpha_h^+{}_h\alpha_i^+{}_i\alpha_h^+ \xmapsto{s_4} (\gamma_i - e_i)(\gamma_i + e_i)^{n-1}{}_i\alpha_h^+ \xmapsto{s_1} {}_i\alpha_h^+(\gamma_h - e_h)(\gamma_h + e_h)^{n-1}$$

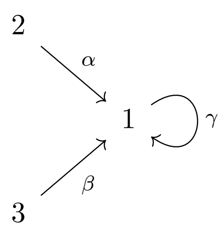
□

3.6 Example of silting objects

According to [AI12](Corollary 2.28), all silting objects in a Krull-Schmidt triangulated category $\mathcal{T}$ have the same number of indecomposable summands. We now give an example which illustrates how this may fail when $\mathcal{T}$ is not Krull-Schmidt.

3.6.1 From 3 generators to 2

Let Λ be the pinched gentle algebra given by



with $\gamma\alpha = \alpha$ and $\gamma\beta = -\beta$. We show that $T := P_1 \oplus T_2$ with $T_2 = (P_2 \oplus P_3 \xrightarrow{(\alpha\ \beta)} P_1)$ is a silting object of $per(\Lambda)^\natural$. We use the embedding of $per(\Lambda)$ into its split closure to identify objects X with couples (X, id_X). Recall that by Lemma 3.5.2 (and the proof of Theorem 3.5.3), $thick(P_1 \oplus P_2 \oplus P_3)$ is $per(\Lambda)^\natural$. For all $x, y \in K$, the commutative diagrams of Figure 3.6.1 show that T does not have positive extensions with itself (here we identify an idempotent e_i with 1).

Moreover, taking the cone of the morphism from P_1 to $P_2 \oplus P_3 \xrightarrow{(\alpha\ \beta)} P_1$ given by the identity of P_1 gives $P_2 \oplus P_3$, which shows that $thick(T)$ is $per(\Lambda)^\natural$. In fact, since there is no morphism from P_1 to $P_2 \oplus P_3$, we can see that T is a tilting object.

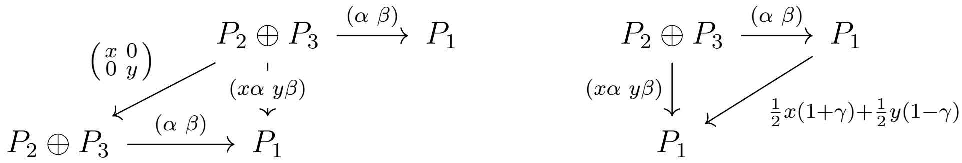
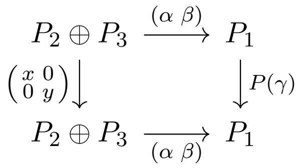


Figure 3.6.1: Any positive self-extension of T is null-homotopic.

We now compute the endomorphism ring of T. First the endomorphism ring of P_1 is given by $\mathrm{End}(P_1) = \{P(\gamma) \mid P \in K[X]\}$. Since there is no morphism from $P1$ to $P_2 \oplus P_3$, $\mathrm{Hom}(P_1, T_2)$ coincides with $\mathrm{End}(P_1) = \{P(\gamma) \mid P \in K[X]\}$. Note that for a polynomial $P \in K[X]$, $P(\gamma)\alpha = P(1)\alpha$ and $P(\gamma)\beta = P(-1)\beta$. Morphims from T_2 to P_1 are given by polynomials P satisfying $P(\gamma)(\alpha\ \beta) = (0\ 0)$, thus the space of morphisms is $\mathrm{Hom}(T_2, P_1) = \{P(\gamma)(\gamma^2 - e_1) \mid P \in K[X]\}$. $\mathrm{End}(T_2)$ can also be identified with $K[X]$ since any morphism



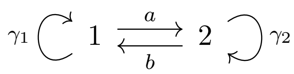
must satisfy $x = P(1)$ and $y = P(-1)$, and thus P determines x and y. Let

$$a := e_1 = (0, e_1) \in \mathrm{Hom}(P_1, T_2),\ b := \gamma^2 - e_1 = (0, \gamma^2 - e_1) \in \mathrm{Hom}(T_2, P_1),$$
$$\gamma_1 := \gamma \in \mathrm{End}(P_1) \text{ and } \gamma_2 := \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \gamma\right) \in \mathrm{End}(T_2).$$

Here we use both the notation $(0, f)$ and f for a morphism from P_1 to T_2 (or from T_2 to P_1). The following relations hold:

$$a\gamma_1 = e_1\gamma = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \gamma\right)(0, e_1) = \gamma_2 a,$$
$$b\gamma_2 = (0, \gamma^2 - e_1)\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \gamma\right) = (0, (\gamma^2 - e_1)\gamma) = \gamma(\gamma^2 - e_1) = \gamma_1 b,$$
$$ba = (\gamma^2 - e_1)e_1 = \gamma_1^2 - e_1,$$
$$ab = (0, e_1)(0, \gamma^2 - e_1) = (0, \gamma^2 - e_1) = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \gamma\right)^2 - \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), e_1\right) = \gamma_2^2 - id_{T_2}.$$

Re-labeling P_1 by 1 and T by 2 gives the following pinched gentle algebra:



with $a\gamma_1 = \gamma_2 a$, $b\gamma_2 = \gamma_1 b$, $ba = \gamma_1^2 - e_1$ and $ab = \gamma_2^2 - e_2$.

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