On algebraic and geometric aspects of the category of projective presentations

Sur des aspects algébriques et géométriques de la catégorie des présentations projectives

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Résumé:

Cette thèse concerne des aspects algébriques et géométriques de la catégorie des présentations projectives sur une algèbre de dimension finie, inspirée à la fois par la théorie de la semistabilité pour les modules au sens de A. King, et la théorie du τ-basculement, telle qu'introduite par A. Adachi, O. Iyama et I. Reiten.

Après un survol des propriétés fondamentales de la catégorie des présentations projectives dans le premier chapitre, nous nous intéressons dans le deuxième chapitre à l'étude des semi-invariants déterminantaux sur les espaces virtuels de présentations projectives. Nous proposons une définition de semistabilité pour les présentations projectives basée sur ces semi-invariants. Bien que cette définition implique une condition numérique sur les g-vecteurs, nous démontrons que les deux concepts ne sont pas équivalents. De plus, nous montrons que la condition numérique sur les g-vecteurs ne capture pas les propriétés homologiques associées à la condition numérique de King pour la semistabilité des modules.

Dans le troisième chapitre, nous considérons des sous-catégories épaisses de présentations projectives, qui incluent les catégories de présentations projectives semistables telles que définies dans le premier chapitre. Nous établissons une correspondance entre les sous-catégories épaisses avec suffisamment d'objets injectifs, les paires de cotorsion complètes et les complexes bousculants à 2-termes de la catégorie des présentations projectives. Cette correspondance fournit un analogue à la bijection entre les sous-catégories vastes finies à gauche, les classes de torsion fonctoriellement finies et les paires à support τ-basculantes dans la catégorie des modules.

Dans le quatrième chapitre, nous abordons le cas où l'algèbre est g-finie au sens de L. Demonet, O. Iyama et G. Jasso. Nous fournissons des équivalences entre la g-finitude, la finitude du nombre de paires de cotorsion, la complétude de toutes les paires de cotorsion et la finitude du nombre de sous-catégories épaisses. De plus, nous montrons que lorsqu'une algèbre satisfait à ces propriétés équivalentes, toutes les sous-catégories épaisses possèdent suffisamment d'objets injectifs.

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Keywords: Representation theory, Homological algebra, Projective presentations, Extriangulated category, τ-tilting theory, Semistability.

Abstract:

This dissertation concerns algebraic and geometric aspects of the category of projective presentations over a finite dimensional algebra, inspired both by the theory of semistability for modules in the sense of A. King, and τ-tilting theory, as introduced by A. Adachi, O. Iyama and I. Reiten.

After recalling the basic properties of the category of projective presentations in the first chapter, in chapter two we delve into the study of determinantal semi-invariants over virtual spaces of projective presentations. We propose a definition of semistability for projective presentations based on these semi-invariants. While this definition implies a numerical condition on g-vectors, we demonstrate that the two concepts are not equivalent. Furthermore, we show that the numerical condition on g-vectors does not capture the homological properties associated with King's numerical condition for module semistability.

In the third chapter we consider thick subcategories of projective presentations, which encompass the categories of semistable projective presentations as defined in the first chapter. We establish a correspondence between thick subcategories with enough injective objects, complete cotorsion pairs and 2-term presilting complex in the category projective presentations. This correspondence provides an analog to the bijection among left finite wide subcategories, functorially finite torsion classes and support τ-tilting pairs in the module category.

In chapter four, we tackle the case when the algebra is g-finite in the sense of L. Demonet, O. Iyama and G. Jasso. We provide equivalences between and algebra being g-finite, having finitely many cotorsion pairs, all cotorsion pairs being complete and having finitely many thick subcategories. Additionally, we show that when an algebra satisfies these equivalent properties, all thick subcategories posses enough injective objects.

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Contents

Introduction en français

Contexte

Théorie des représentation des algèbres de dimension finie

Cette thèse s'inscrit dans le domaine de la théorie des représentations des algèbres de dimension finie. Soit $\Bbbk$ un corps, une $\Bbbk$-*algèbre* est un $\Bbbk$-espace vectoriel muni d'une multiplication compatible avec la structure de l'espace vectoriel. Une algèbre est dite de *dimension finie* si sa dimension en tant que $\Bbbk$-espace vectoriel est finie. Des exemples classiques de telles structures sont les algèbres de matrices. Considérons, par exemple, l'ensemble des matrices diagonales inférieures de taille 2×2 sur un corps $\Bbbk$: $\Lambda_{A_2} = \left\{ \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix} \right) \mid a, b, c \in \Bbbk \right\}$, alors Λ_{A_2} est un $\Bbbk$-espace vectoriel de dimension finie (de dimension 3) muni d'une multiplication (multiplication matricielle standard) compatible avec l'addition et la multiplication scalaire, définissant ainsi une $\Bbbk$-algèbre. L'algèbre Λ_{A_2} peut être décrite via la combinatoire d'un graphe orienté fini, ou *carquois.* En effet, supposons que A_2 soit le graphe avec deux *sommets* et une *flèche*

$$1 \xrightarrow[\alpha]{} 2\,.$$

A A_2 nous pouvons associer l'espace vectoriel $\Bbbk A_2 = \Bbbk e_1 \oplus \Bbbk e_2 \oplus \Bbbk\alpha$, où e_i est le chemin correspondant au sommet i pour $i = 1, 2$, et α est le chemin qui suit la seule flèche allant de 2 à 1. En dotant $\Bbbk A_2$ du produit induit par la concaténation des chemins, nous donnons à $\Bbbk A_2$ la structure d'une $\Bbbk$-algèbre. L'algèbre $\Bbbk A_2$ est un exemple *d'algèbre de chemins* sur un carquois. De plus, l'application donnée par $e_1 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$, $e_2 \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)$, $\alpha \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right)$ induit un isomorphisme d'algèbres entre Λ_{A_2} et $\Bbbk A_2$. Lorsque $\Bbbk$ est un corps algébriquement clos, tout $\Bbbk$-algèbre *basique* de dimension finie est isomorphe au quotient d'une algèbre de chemins $\Bbbk Q_\Lambda$ par un idéal I_Λ de *relations admissibles.* Ce fait a été observé pour la première fois par P. Gabriel au début des années 70 [Gab72], et depuis les carquois sont devenus un outil essentiel dans l'étude des algèbres de dimension finie. Un *module* (à droite) sur une $\Bbbk$-algèbre de dimension finie Λ est un $\Bbbk$-espace vectoriel muni d'une action (à droite) de Λ. La collection de tous les Λ-modules ainsi que les morphismes Λ-linéaires entre eux est un exemple de *catégorie abélienne*, notée $\operatorname{mod}\Lambda$. Lorsque Λ est l'algèbre des chemins $\Bbbk Q/I$ associée à un carquois à relations (Q, I), la catégorie $\operatorname{mod}\Lambda$ est équivalente à la catégorie des *représentations* $\operatorname{rep}(Q^{op}, I^{op})$ du carquois à relations opposé (Q^{op}, I^{op}). Une *représentation* d'un carquois à relations (Q, I) associe un espace vectoriel de dimension finie à chaque sommet et d'une application linéaire à chaque flèche, de telle sorte que la composition des applications linéaires correspondant aux relations dans I est nulle.

Semistabilité et présentations projectives

La catégorie $\operatorname{mod}\Lambda$ possède la particularité que chaque Λ-module peut être décomposé en une somme directe unique (à isomorphisme pèrs) d'un nombre fini de modules *indécomposables*, c'est-à-dire, de modules qui ne peuvent pas être décomposés en une somme directe de deux sous-modules propres. L'un des théorèmes fondamentaux de la théorie des représentations des algèbres de dimension finie est la dichotomie *docile-sauvage* [Dro06, DG92], qui implique que la classification des modules indécomposables dans $\operatorname{mod}\Lambda$ est un problème intraitable et ne peut pas être traitée uniquement à l'aide d'outils de dénombrement. Lorsque les méthodes de comptage traditionnelles deviennent insuffisantes, les mathématiciens se tournent souvent vers l'analyse de la géométrie du problème. En théorie des représentations, cette approche a conduit à l'introduction de *l'espace de modules des représentations* (cf. [Kin94, Rei08]), une variété algébrique dont les points paramètrent des portions de la catégorie des représentations. Etant donné un carquois à relations (Q, I), l'ensemble des représentations de même *vecteur de dimension* forme une variété dotée d'une action de groupe algébrique dont les orbites sont constituées de représentations isomorphes. Dans ce cadre, le problème de la classification de toutes les représentations de (Q, I) peut être considéré comme l'étude des ensembles d'orbites pour chaque vecteur de dimension. Les ensembles de toutes les orbites sont rarement des variétés algébriques, mais des sous-ensembles des orbites peuvent être considérés. En ce qui concerne le choix des orbites, D. Mumford a fourni une réponse pour les variétés quasi-projectives sous l'action de groupes algébriques réductifs : il suffit de prendre les orbites des points qui sont *θ-semistable* (Définition 2.2, [MFK94]). Les espaces de modules des représentations sont les variétés d'orbites de représentations θ-semistables (à une relation d'équivalence près). En théorie des représentations, ces θ peuvent être codés avec des vecteurs entiers et sont appelés *conditions de stabilité.*

$$\begin{tikzcd}
 & & & 4 \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ur] \arrow[dr] & \\
 & & & 5
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & \mathbb{C} \\
\mathbb{C} \arrow[r, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & \mathbb{C}^2 \arrow[r, "(1,-1)"] & \mathbb{C} \arrow[ur, "1"] \arrow[dr, "0"'] & \\
 & & & 0
\end{tikzcd}$$

Figure 1: Exemple d'un carquois et d'une représentation. Le vecteur de dimension de cette représentation est $(1, 2, 1, 1, 0)$.

Depuis son introduction dans les années 1990, la semistabilité a occupé un rôle central dans la théorie des représentations. Un résultat illustrant son importance est celui obtenu par C. Ingalls et H. Thomas dans [IT09]. Ils ont montré que les sous-catégories *finies à gauche* des représentations semistables associées à des carquois héréditaires Q correspondent à divers objets de la catégorie des représentations, tels que les *paires de torsion fonctoriellement finies* et les *paires à support τ-basculentes*, ainsi que des objets combinatoires associés à Q comme les *partitions non croisées* (pour les carquois *Dynkin*) et les amas dans l'*algèbre amassée* associée à Q [IT09, Théorème 1. 1]. La relation entre ces différents aspects a continué à se développer et constitue toujours une ligne de recherche très active en théorie des représentations [BMR$^+$06, AIR14, MŠ17, Yur18, BST19, DIJ19, BH23, AP22, BH23, BDM$^+$20, STW16].

Géométriquement, une représentation est θ-semistable s'il existe une fonction régulière, appelée *θ-semi-invariant* ou semi-invariant de *poids θ*, qui ne s'annule pas lorsqu'elle est évaluée sur le point associé à la représentation. Un ingrédient clé dans la construction de ces fonctions régulières est l'utilisation de *présentations projectives* [AB69, AR73, Pau08, PZ23]. Une présentation projective d'un Λ-module donné M est une application $\begin{smallmatrix} X_M^{-1} \\ \downarrow x_M \\ X_M^0 \end{smallmatrix}$ entre modules *projectifs* X_M^{-1} et X_M^0 telle que le conoyau de x_M est isomorphe à M. Lorsque Λ est une algèbre de dimension finie, tous les modules de $\operatorname{mod}\Lambda$ admettent une présentation projective. Étant donné deux présentations projectives $\begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^0 \end{smallmatrix}$ et $\begin{smallmatrix} Y^{-1} \\ \downarrow y \\ Y^0 \end{smallmatrix}$, un morphisme entre elles est donné par un carré commutatif

$$\begin{tikzcd} X^{-1} \arrow[r, "f^{-1}"] \arrow[d, "x"] & Y^{-1} \arrow[d, "y"] \\ X^0 \arrow[r, "f^0"] & Y^0. \end{tikzcd}$$

La collection de toutes les présentations projectives, ou *complexes à 2 termes*, et leurs morphismes à *homotopie* près, forme une *catégorie extriangulée*, qu'on note $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ (Définition 1.8, [NP19]). Comme pour les représentations et leurs vecteurs de dimension, nous pouvons classer les présentations projectives à l'aide d'un vecteur entier appelé *g-vecteur*, qui désigne la classe d'une présentation projective dans le groupe de Grothendieck $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$. Étant donné un carquois et une condition de stabilité θ, tous les θ-semi-invariants sont engendrés par certains polynômes $s(X,-)$, appelées *semi-invariants déterminantaux* (Définition 2.6), introduits pour la première fois par A. Schofield dans [Sch91]. Ces polynômes sont construits en utilisant des présentations projectives X dont les g-vecteurs sont multiples positif de θ [DW00, SVdB01, Dom02]. Ainsi, pour vérifier si une représentation M est θ-semistable, il suffit de trouver une présentation projective X dont le vecteur g est un multiple positif de θ tel que $s(X,M)$ soit non nul. En utilisant le critère de Hilbert-Mumford, A. King a montré que le fait d'être θ-semistable possède une interprétation algébrique directe [Kin94]. Il a montré qu'une représentation M est θ-semistable si une relation numérique entre son vecteur de dimension et la condition de stabilité θ est satisfaite (Proposition 2.4) : le vecteur θ doit être orthogonal au vecteur de dimension de M, et pour toute sous-représentation non triviale $N \subset M$, le produit scalaire entre θ et le vecteur de dimension de N doit être inférieur ou égal à zéro.

Dualité tropicale et semistabilité

Des exemples importants de conditions de stabilité sont ceux donnés par les g-vecteurs des complexes *bousculants* à 2-termes (Définition 3.27). En effet, Les catégories de modules semistables qui leur sont associées sont précisément celles qui sont en correspondance avec les classes de torsion et les paires à support τ-basculantes. Elles jouent également un rôle crucial dans la catégorification des *algèbres amassées* [FZ02]. Lorsque Λ est l'algèbre de Jacobi d'un carquois à potentiel non dégénéré (Q,W), les complexes bousculants à 2-termes *atteignables* sont en bijection avec les *amas* de l'algèbre amassée associée à Q [CIKLFP13]. En particulier, les g-vecteurs des complexes *bousculants* à 2-termes atteignables coïncident avec des vecteurs entiers qui déterminent les variables de amas qui leur sont associées, et sont également appelés g-vecteurs [1] [FZ07, Proposition 2.10].

À chaque amas est associé un autre ensemble de vecteurs entiers appelés *c-vecteurs*, qui correspondent aux exposants des *coefficients* liés aux variables d'amas. L'ensemble des g-vecteurs et l'ensemble des c-vecteurs associés à un amas donné forment une $\mathbb{Z}$-base de $\mathbb{Z}^n$ et sont duaux l'un par rapport à l'autre [NZ12]. Cette dualité, connue sous le nom de *dualité tropicale*, va au-delà d'un simple fait numérique : elle peut être interprétée comme un reflet des interactions entre les catégories $\operatorname{mod}\Lambda$ et $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ lorsque Λ est une algèbre de Jacobi. Par exemple, on sait que les c-vecteurs correspondent aux vecteurs de dimension de certains modules connus sous le nom de *briques* [Nag13, NC13] (Définition 4.23). Dans [Tre19], H. Treffinger a montré que les briques dont les vecteurs de dimension correspondent aux c-vecteurs associés à l'amas d'un complexe bousculant à 2-termes, pouvaient être obtenues en considérant des sous-catégories de représentations semistables pour des conditions de stabilité données par les g-vecteurs associés [Tre19, Theorem 3.5]. En particulier, la dualité entre la base des c-vecteurs et des g-vecteurs d'un complexe bousculant à 2-termes donné découle des conditions numériques définissant la semistabilité. En bref, les g-vecteurs s'avèrent être des conditions de stabilité pour les c-vecteurs. De plus, dans [IOTW09], K. Igusa, K. Orr, G. Todorov, et J. Weyman ont montré que les semi-invariants determinantaux $s(-,M)$ pour M à vecteur de dimension δ orthogonal à un θ donné, définissent des δ-semi-invariants *virtuels* pour les présentations projectives à g-vecteur θ (Section 2.2.1). Par ailleurs, dans un article ultérieur [IOTW15], ils ont établi que lorsque Λ est une algèbre héréditaire de dimension finie, les c-vecteurs associés à un complexe bousculant à 2-termes sont des poids de semi-invariants déterminantaux (à un signe près). Cela soulève la question suivante : les c-vecteurs pourraient-ils être interprétés comme des conditions de stabilité pour les g-vecteurs ?

Sommaire de la thèse et contributions

Dans cette thèse, nous étudions la catégorie des complexes à 2-termes $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ dans le contexte de la semistabilité et de la théorie du τ-basculement. Notre but est d'explorer la possibilité de définir une notion de semistabilité pour les présentations projectives qui reflète à la fois des propriétés géométriques et homologiques semblables à celles pour les représentations.

La thèse est organisée en quatre chapitres. Dans le chapitre 1, nous rappelons la définition d'une catégorie extriangulée et présentons la catégorie homotopique des complexes à 2-termes des modules projectifs $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ sur une algèbre de dimension finie Λ dans ce contexte. Tout au long du texte, nous nous référerons à $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ comme la *catégorie des présentations projectives*. Nous présentons des propriétés classiques de $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, et fournissons des preuves élémentaires pour la plupart d'entre elles. En particulier, nous rappelons que $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ est une *catégorie extriangulée* 0-*Auslander* au sens de [GNP23]. Dans ce premier chapitre, nous introduisons la notion qui est au centre de ce manuscrit : celle de *sous-catégorie épaisse* (Défintion 1.16) d'une catégorie extriangulée, telle que définie dans [NOS22]. Le chapitre 2 est consacré à la notion géométrique de semistabilité et

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[1] Le terme *g-vecteur* pour les classes de complexes à 2-termes dans le groupe de Grothendieck $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$ est dérivé de la théorie des algèbres amassées.

à celle de semi-inivariant déterminantal. Nous montrons que l'existence d'un semi-invariant virtuel non-nul de poids δ sur une variété de présentations projectives implique une condition numérique entre le poids δ et les g-vecteurs des complexes à 2-termes, qui est duale à celle découverte par A. King pour les modules semistables ; nous montrons un cas où la réciproque n'est pas vérifiée. Nous fournissons également un exemple montrant que, contrairement aux sous-catégories de modules qui sont semistables au sens de A. King, les sous-catégories des complexes à 2-termes qui sont *numériquement semistables* pour un certain poids δ, ne définissent pas des sous-catégories intéressantes de $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Nous proposons à la place une autre notion, celle de *M-semistabilité* pour $M \in \operatorname{mod}\Lambda$ (Définitions 2.15 et 3.66), qui est définie à l'aide des semi-invariants déterminantaux.

Les sous-catégories de modules M-semistables sont un exemple de sous-catégories épaisses de $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Dans le chapitre 3, après avoir revisité la théorie du τ-basculement selon T. Adachi, O. Iyama, et I. Reiten, ainsi que les correspondances d'Ingalls et Thomas, nous montrons que les sous-catégories épaisses avec *suffisamment d'injectifs* sont en correspondance avec les *paires de cotorsion* et les complexes bousculants à 2-termes. Nous montrons que ces nouvelles bijections reflètent celles entre les sous-catégories vastes, les paires de torsion et les paires à support τ-basculantes découvertes pour la première fois par C. Ingalls et H. Thomas. Au chapitre 4, nous étudions le cas où Λ est g-fini, c'est-à-dire le cas où Λ possède un nombre fini de classes d'isomorphisme de complexes bousculants à 2 termes. L. Demonet, O. Iyama et G. Jasso ont montré qu'une algèbre est g-finie si et seulement si toutes les classes de torsion sont fonctoriellement finies ; ou de manière équivalente, si et seulement s'il existe un nombre fini de sous-catégories vastes[2]. Nous montrons qu'un même énoncé vaut pour $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, à savoir que Λ est g-finie si et seulement si toutes les classes de cotorsion sont complètes, et si et seulement s'il existe un nombre fini de sous-catégories épaisses. Enfin, nous montrons que lorsque Λ est une algèbre g-finie, toutes les sous-catégories épaisses ont suffisamment d'injectifs.

Les résultats des chapitres 2 et 3 ont été soumis pour publication et sont disponibles sur [Gar23]. Ceux du chapitre 4 ont été présentés lors du workshop *Cluster Algebras and Its Applications* à Oberwolfach en janvier 2024.

Stabilité numérique pour les présentations projectives

Soit Λ une $\mathbb{C}$-algèbre basique de dimension finie, et soit $n = |\Lambda|$ le nombre de ses facteurs directs indécomposables. Considérons $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ la sous-catégorie pleine de complexes à 2-termes de la catégorie homotopique des complexes de projectives bornées $\mathcal{K}^b(\operatorname{proj}\Lambda)$. Pour tout $X = \begin{array}{c} X^{-1} \\ \downarrow x \\ X^0 \end{array}$ on note $[X]$ la classe de X dans le groupe de Grothendieck $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$. Rappelons que $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) = \bigoplus_{i=1}^n \mathbb{Z}[P_i]$, où les P_i sont les n facteurs directs indécomposables de Λ. La classe $[X]$ s'identifie avec le vecteur

$$[X^0] - [X^{-1}] = \sum_{i=1}^{n} \theta_i^0 [P_i] - \sum_{j=1}^{n} \theta_j^{-1} [P_j]$$

---

[2] Leur bijection est énoncée en termes de *briques*, qui correspondent à des objets simples de sous-catégories vastes de $\operatorname{mod}\Lambda$.

où θ_i^{-1} et θ_i^0 satisfont à $X^{-1} \simeq \bigoplus_{i=1}^n P_i^{\oplus\theta_i^{-1}}$ et $X^0 \simeq \bigoplus_{i=1}^n P_i^{\oplus\theta_i^0}$. On rappelle l'existence d'un accouplement $\langle -,- \rangle$ entre $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$ et $K_0(\operatorname{mod}\Lambda)$ tel que pour $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ et $M \in \operatorname{mod}\Lambda$

$$\langle [X],[M] \rangle = \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^0,M)) - \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^{-1},M)).$$

À tout $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$, on associe son *espace virtuel de présentations* (Définition 2.16) tel qu'introduit dans [IOTW09]:

$$R^{vir}(\theta) = \varinjlim_{\gamma\in\mathbb{Z}^n_{\geq 0}} \operatorname{Hom}_\Lambda(P^{\theta^-} \oplus P^\gamma, P^{\theta^+} \oplus P^\gamma)$$

où $\theta^- = -(\min\{0,\theta_i\})_{1\leq i\leq n}$, $\theta^+ = (\max\{0,\theta_i\})_{1\leq i\leq n}$ et $P^\eta = \bigoplus_{i=1}^n P^{\oplus\eta_i}$ pour tout $\eta \in \mathbb{Z}^n_{\geq 0}$. Un *semi-invariant virtuel* est un élément de l'anneau

$$SI^{vir}(\theta) = \varprojlim_{\gamma\in\mathbb{Z}^n_{\geq 0}} SI(\operatorname{Hom}_\Lambda(P^{\theta^-} \oplus P^\gamma, P^{\theta^+} \oplus P^\gamma))^{\operatorname{Aut}_\Lambda(P^{\theta^-}\oplus P^\gamma)\times\operatorname{Aut}_\Lambda(P^{\theta^+}\oplus P^\gamma)},$$

où $SI(\operatorname{Hom}_\Lambda(P^{\theta^-} \oplus P^\gamma, P^{\theta^+} \oplus P^\gamma)^{\operatorname{Aut}_\Lambda(P^{\theta^-}\oplus P^\gamma)\times\operatorname{Aut}_\Lambda(P^{\theta^+}\oplus P^\gamma)}$ désigne l'anneau de semi-invariants du $\Bbbk$-espace vectoriel $\operatorname{Hom}_\Lambda(P^{\theta^-} \oplus P^\gamma, P^{\theta^+} \oplus P^\gamma)$ sous l'action du groupe algébrique $\operatorname{Aut}_\Lambda(P^{\theta^-} \oplus P^\gamma) \times \operatorname{Aut}_\Lambda(P^{\theta^+} \oplus P^\gamma)$.

Proposition ([IOTW09, Proposition 5.13])**.** *Soit* $X^{-1}, X^0 \in \operatorname{proj}\Lambda$ *et* $M \in \operatorname{mod}\Lambda$*, tels que* $\langle [X^0]-[X^{-1}],[M] \rangle = 0$*. Alors la fonction régulière*

$$s(x,M) = \det\left(\operatorname{Hom}_\Lambda(X^0,M) \xrightarrow{\operatorname{Hom}_\Lambda(x,M)} \operatorname{Hom}_\Lambda(X^{-1},M)\right)$$

est un $\operatorname{Aut}_\Lambda(X^{-1})\times\operatorname{Aut}_\Lambda(X^0)$*-semi-invariant de* $\operatorname{Hom}_\Lambda(X^{-1},X^0)$ *de poids* $([M],[M]) \in \mathbb{Z}^{2n}_{\geq 0}$*. En plus, il correspond à un semi-invariant virtuel dans* $SI^{vir}(\theta)$ *pour* $\theta = [X^0]-[X^{-1}]$.

Les fonctions régulières $s(-,-)$ sont appelées *semi-invariants déterminantaux.* Elles ont été introduites pour la première fois par A. Schofield dans [Sch91], puis il a été montré qu'elles engendrent les anneaux des semi-invariants des variétés de représentations [SVdB01]. En particulier, pour vérifier si un module est θ-semistable, il suffit de trouver $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ et $m \in \mathbb{Z}_{\geq 0}$ tels que $[X] = m\theta$ et $s(X,M) \neq 0$. Cela inspire la définition suivante.

Définition (**2.15**, M**-stabilité**)**.** *Soit* $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *et* $M \in \operatorname{mod}\Lambda$*. Le complexe* X *est* M-semistable *si et seulement si*

i) $\langle [X],[M] \rangle = 0$*;*

ii) $s(X,M) \neq 0$.

Comme mentionné dans la section précédente, lors de l'analyse de la semistabilité des modules, on pourrait entièrement éviter la géométrie et déterminer si un module est θ-semistable pour un θ donné dans $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$ en vérifiant une condition numérique simple :

Proposition ([Kin94, Proposition 3.1])**.** *Soit* $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$*, alors* $M \in \operatorname{mod}\Lambda$ *est (géométriquement)* χ_θ*-semistable si et seulement si* *i)* $\langle \theta, [M] \rangle = 0$*;*

ii) Pour tout sous-module $N \subset M$*,* $\langle \theta, [N] \rangle \leq 0$.

De plus, la sous-catégorie $\mathscr{W}_\theta$ *des modules* θ*-semistables est une sous-catégorie abélienne de* $\operatorname{mod}\Lambda$.

Un morphisme $f : Y \to X$ dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ est dit une *inflation* s'il existe un triangle $Y \xrightarrow{f} X \to Z \dashrightarrow Y[1]$ tel que $Z \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. La proposition précédente conduit à la définition suivante.

Définition (2.21, Semistabilité numérique). *Soit* $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *et* $d \in \mathcal{K}_0(\operatorname{mod}\Lambda)$*. On dit que* X *est* numériquement d-semistable *si*

i) $\langle [X], d \rangle = 0$*,*

ii) Pour toute inflation $Y \rightarrowtail X$*, on a que* $\langle [Y], d \rangle \geq 0$.

Le résultat principal de la Section 2 établit la relation entre la semistabilité numérique et la M-semistabilité.

Proposition (2.22). *Soit* $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *et* $M \in \operatorname{mod}\Lambda$ *dont la classe dans* $\mathcal{K}_0(\operatorname{mod}\Lambda)$ *est* $[M]$*. Si* X *est* M*-semistable, alors* X *est numériquement* $[M]$*-semistable.*

Nous fournissons un exemple où la réciproque de la Proposition 2.22 est fausse. En effet, dans l'exemple 2.24, nous présentons une algèbre de chemins non héréditaire Λ pour laquelle nous pouvons trouver $\delta \in K_0(\operatorname{mod}\Lambda)$ et $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfaisant que X est numériquement δ-semistable, mais tel qu'il n'existe pas de semi-invariant f de poids δ (en particulier pas de semi-invariant déterminantal), sur $\operatorname{Hom}_\Lambda(X^{-1} \oplus P^\gamma, X^0 \oplus P^\gamma)$ pour tout $\gamma \in \mathbb{Z}^n_{\geq 0}$ tel que $f(X) \neq 0$. Enfin, nous montrons que les catégories de présentations projectives numériquement δ-semistables ne sont pas fermées par extensions dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ en général, ce qui met en évidence les limites de définir de façon directe une condition numérique de semistabilité pour les présentations projectives duale à celle de King pour les modules.

Correspondances d'Ingalls-Thomas pour les présentations projectives

Étant donné une condition de stabilité $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$, la sous-catégorie pleine $\mathscr{W}_\theta$ des modules qui sont θ-semi-stables forme une sous-catégorie abélienne, ou *vaste*, de $\operatorname{mod}\Lambda$. Outre leur importance géométrique, les sous-catégories de modules semistables sont pertinentes en théorie des représentation en raison de leurs liens avec la *théorie du* τ*-basculement.* Cette connexion a été mise en lumière pour la première fois par C. Ingalls et H. Thomas dans [IT09], qui ont établi des correspondances entre les ensembles suivants :

1) l'ensemble $\mathrm{s}\tau\text{-tilt}\,\Lambda$ de classes d'isomorphisme de paires à support τ-basculantes basiques [3] (Définition 3.2),

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[3] Dans leur article original, cette correspondance est établie en termes des *objets amas-basculants*, qui sont en bijection avec les paires à support τ-basculantes [AIR14].

2) l'ensemble $\operatorname{f-tors}\Lambda$ de classes de torsion fonctoriellement finies dans $\operatorname{mod}\Lambda$ (Définition 3.3),

3) l'ensemble $\operatorname{l-wide}\Lambda$ de catégories vastes finies à gauche $\operatorname{mod}\Lambda$ (Définition 3.20),

4) l'ensemble des sous-catégories semistables finies à gauche de $\operatorname{mod}\Lambda$,

lorsque Λ est une algèbre *héréditaire*. Les bijections entre 1), 2) et 3) pour toute algèbre Λ ont été démontrées par F. Marks et J. Šťovíček dans [MŠ17]. La généralisation de la bijection entre 3) et 4) a été obtenue indépendamment par T. Yurikusa dans [Yur18] et T. Brüstle, D. Smith et H. Treffinger dans [BST19]. En fait, toutes les sous-catégories vastes finies à gauche peuvent être réalisées en tant que sous-catégories de modules pour lesquels le semi-invariant déterminantal $s(X,-)$ ne s'annule pas pour certains complexes bousculants à 2-termes $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$:

Théorème (2.10, [Yur18, BST19]). *Pour tout sous-catégorie finie à gauche $\mathcal{W}$, il existe un complexe bousculant à 2-termes U dont le g-vecteur est θ tel que*

$$\mathcal{W} = \mathscr{W}_\theta = \mathscr{W}(U) = \{M \in \operatorname{mod}\Lambda \mid \langle \theta, [M]\rangle = 0 \text{ et } s(U,M) \neq 0\}.$$

Les semi-invariants déterminantaux définissent également des sous-catégories de $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ présentant des propriétés intéressantes :

Proposition (3.70). *Soit $M \in \operatorname{mod}\Lambda$, alors la sous-catégorie pleine*

$$\mathscr{T}(M) = \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid s(X,M) \neq 0\}$$

est épaisse, c'est-à-dire, qu'elle est additive et satisfait que pour tout triangle $X \to Y \to Z \dashrightarrow X[1]$ avec $X, Y, Z \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, si deux parmi les trois objets sont dans $\mathscr{T}(M)$, alors le troisième l'est aussi. Autrement dit, la sous-catégorie $\mathscr{T}(M)$ est fermée par extensions, cônes et cocônes dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Inspirées par la proposition précédente, nous introduisons de nouvelles bijections entre les sous-catégories épaisses et autres classes d'objets étudiées dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, à savoir les *paires cotorsion complètes* (Définition 3.40) et les complexes bousculants à 2-termes. Ces nouvelles correspondances reflètent celles entre les sous-catégories vastes finies à gauche, les classes de torsion fonctoriellement finies et les paires à support τ-basculantes.

Théorème (3.51). *Soit Λ une $\Bbbk$-algèbre de dimension finie. Il existe des application bien définies*

$$\begin{tikzcd}\operatorname{cotor}\Lambda \arrow[r, shift left, "\beta"] & \operatorname{thick}\Lambda \arrow[l, shift left, "I"]\end{tikzcd}$$

entre l'ensemble $\operatorname{c-cotor}\Lambda$ *de paires de cotorsion et l'ensemble* $\operatorname{thick}\Lambda$ *de sous-catégories épaisses dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Elles sont données par*

$$\beta((\mathcal{X},\mathcal{Y})) = \{X \in \mathcal{X} \mid \forall\ X \to X' \to X'' \dashrightarrow X[1] \text{ tel que } X' \in \mathcal{X},\ \text{ alors } X'' \in \mathcal{X}\}$$

$$I(\mathcal{T}) = \left({}^{\perp_1}(\mathcal{T}^{\perp_1}), \mathcal{T}^{\perp_1}\right),$$

pour $(\mathcal{X},\mathcal{Y}) \in \operatorname{cotor}\Lambda$ et $\mathcal{T} \in \operatorname{thick}\Lambda$. De plus, quand on les restreint aux ensembles $\operatorname{c-cotor}\Lambda$ *de paires de cotorsion complètes et* $\operatorname{inj-thick}\Lambda$ *des sous-catégories épaisses avec suffisament d'injectifs, β et I sont inverses l'une de l'autre.*

Théorème (3.72). *Soit Λ une $\Bbbk$-algèbre de dimension finie. Il existe des application bien définies*

$$\begin{tikzcd} \operatorname{wide}\Lambda \arrow[r, shift left, "\mathscr{T}"] & \operatorname{thick}\Lambda \arrow[l, shift left, "\mathscr{W}"] \end{tikzcd}$$

telles que pour tout $\mathcal{W} \in \operatorname{wide}\Lambda$ et $\mathcal{T} \in \operatorname{thick}\Lambda$

$$\begin{aligned} \mathscr{T}(\mathcal{W}) &= \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid X \textit{ is } M\textit{-semistable } \forall\ M \in \mathcal{W}\} \\ \mathscr{W}(\mathcal{T}) &= \{M \in \operatorname{mod}\Lambda \mid X \textit{ is } M\textit{-semistable } \forall\ X \in \mathcal{T}\}. \end{aligned}$$

De plus, l'application $\mathscr{W}$ donne une bijection entre l'ensemble de sous-catégories avec suffisamment d'injectifs et l'ensemble de sous-catégories vastes à gauche.

Corollaire (3.33). *Il existe des bijections entre*

1) Classes d'isomorphisme de complexes bousculants à 2-termes dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

2) Paires de cotorsion complètes dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

3) Sous-catégories épaisses dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ avec suffisamment d'injectifs.

Ces bijections sont compatibles avec celles entre les paires à support τ-basculantes, les classes de torsion fonctoriellement finies et les sous-catégories vastes finies à gauche ; en d'autres termes, le diagramme suivant est commutatif :

$$\begin{tikzcd}
 & & 2\text{-silt}\,\Lambda \arrow[dl, "\Psi"', "\text{[AT22]}"] \arrow[dr, "\operatorname{thick}(U_\rho)"] \arrow[ddd, "H^0"', "\text{[AIR14]}"] & & \\
 & \text{c-cotor}\,\Lambda \arrow[rr, "\beta"] \arrow[ddd, "\Phi"', "\text{[PZ23]}"] & & \text{inj-thick}\,\Lambda \arrow[ddd, "\mathscr{W}"] & \\
\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) & & & & \\
 & & s\tau\text{-tilt}\,\Lambda \arrow[dl, "\operatorname{Fac}"', "\text{[AIR14]}"] \arrow[dr] & & \operatorname{mod}\Lambda \\
 & \text{f-tors}\,\Lambda \arrow[rr, "\alpha", "\text{[MŠ17]}"'] & & \text{l-wide}\,\Lambda &
\end{tikzcd}$$

Sous-catégories épaisses des algèbres g-finies

Une algèbre Λ est dite *g-finie* si elle a un nombre fini de classes d'isomorphisme de paires à support τ-basculantes ou, de manière équivalente, si elle admet un nombre fini de classes d'isomorphisme de complexes bousculants à 2-termes. Cette notion a été introduite et étudiée en détail par L. Demonet, O. Iyama et G. Jasso dans [DIJ19], où ils établissent le résultat suivant :

Théorème (4.2). *[DIJ19, Théorèmes 3.8 et 4.2] Soit Λ une $\Bbbk$-algèbre de dimension finie. Les énoncés suivants sont équivalents :*

1) Λ *es g-finie.*

2) Il existe un nombre fini de paires de torsion fonctoriellement finies $\operatorname{mod}\Lambda$.

3) Toutes les classes de torsion dans $\operatorname{mod}\Lambda$ *sont fonctoriellement finies.*

4) Il existe un nombre fini de briques dans $\operatorname{mod}\Lambda$.

Le corollaire suivant suit de la définition d'une sous-catégorie vaste finie à gauche et du théorème précèdent.

Corollaire (4.3). *Soit* Λ *une* $\Bbbk$*-algèbre de dimension finie. Si* Λ *est g-finie, toutes les catégories vastes sont finies à gauche.*

Dans le chapitre 4, nous étudions les homologues du Théorème 4.2 et du Corollaire 4.3 dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Un outil clé utilisé dans les démonstrations de nos résultats est la *réduction* de $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ par rapport à un complexe bousculant U [GNP23]. On note T_U la complétion de Bongartz de U, C_U l'algèbre dg (*différentielle graduée*) $\mathcal{E}\mathrm{nd}_b(T_U)/\langle e_U\rangle$ où e_U est l'idempotent associé à U, et $\operatorname{per}(C_U)$ la catégorie des complexes parfaits sur C_U associée.

Lemme (4.43). *Soit* $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *un complexe bousculant à* 2*-termes et soit* $\mathcal{U} = \operatorname{add}(U)$*. Soit* $\mathcal{Z}_{\mathcal{U}} = \{X \in \mathcal{K}^b(\operatorname{proj}\Lambda) \mid \operatorname{Hom}_b(X, U[i]) = 0 = \operatorname{Hom}_b(U[-i], X)\,\forall i > 0\}$ *et* $\rho : \mathcal{K}^b(\operatorname{proj}\Lambda) \to \mathcal{K}^b(\operatorname{proj}\Lambda)/\operatorname{thick}_b(\mathcal{U})$ *le foncteur de localisation. Si* $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *est une sous-catégorie épaisse avec* $U \in \mathcal{H}$*, alors*

$$\frac{\mathcal{H} \cap \mathcal{Z}_{\mathcal{U}}}{[U]} \simeq \rho(\mathcal{H}).$$

En particulier, $\rho(\mathcal{H})$ *est épaisse dans la catégorie extriangulée*

$$\rho(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) \simeq \operatorname{per}^{[\text{-}1,0]}(C_U) = C_U * C_U[1].$$

Le lemme 4.43 est un des ingrédients principaux dans la preuve du résultat suivant :

Théorème (4.51). *Soit* Λ *une* $\Bbbk$*-algèbre de dimension finie. Soit* $\mathcal{H}$ *une sous-catégorie épaisse de* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$*, alors il existe un complexe bousculant à* 2*-termes* $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *tel que* $\mathcal{H} = thick_{[-1,0]}(U)$.

Les principaux résultats du chapitre 4 fournissent des analogues du Théorème 4.2 et du Corollaire 4.3.

Théorème (4.4). *Soit* Λ *une algèbre* $\Bbbk$*-finie de dimension finie. Les énoncés suivants sont équivalents :*

1) Λ *est g-finie.*

2) Il existe un nombre fini de classes cotorsion complètes dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

3) Toutes les classes cotorsion dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *sont complètes.*

4) Il existe un nombre fini de sous-catégories épaisses dans $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Les équivalences entre les énoncées 1), 2) et 3) découlent du résultat suivant.

Théorème (4.44). *Soit* Λ *une* $\Bbbk$*-algèbre de dimension finie. Alors le foncteur* H^0 : $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \to \Lambda$ *induit une bijection*

$$\begin{aligned} H^0 : \operatorname{cotor}\Lambda &\longrightarrow \operatorname{tors}\Lambda \\ (\mathcal{X},\mathcal{Y}) &\longmapsto H^0(\mathcal{Y}). \end{aligned}$$

On note que la correspondance entre les paires cotorsion et les paires de torsion dans le théorème 4.44 a été introduite pour la première fois par D. Pauksztello et A. Zvonareva dans [PZ23], qui ont montré qu'elle induisait une bijection entre les paires cotorsion complètes et les classes de torsion fonctoriellement finies. Notre contribution est d'avoir étendu la bijection à toutes les paires cotorsion et torsion.

Le dernier résultat dans le chapitre 4 et dans ce manuscrit est l'analogue du corollaire 4.3 et de son dual :

Théorème (4.56). *Soit* Λ *une* $\Bbbk$*-algèbre de dimension finie g-finie, alors toutes les sous-catégories épaisses de* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *ont suffisamment d'injectifs.*

Théorème (4.58). *Soit* Λ *une* $\Bbbk$*-algèbre de dimension finie g-finie, alors toutes les sous-catégories épaisses de* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *ont suffisamment de projectifs.*

Corollaire (4.59). *Soit* Λ *une* $\Bbbk$*-algèbre de dimension finie g-finie, alors toutes les sous-catégories épaisses de* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *ont suffisamment de projectifs et d'injectifs.*

Introduction

Context

Representation theory of finite-dimensional algebras

The subject area of this thesis is the representation theory of finite-dimensional algebras. Let $\Bbbk$ be a field, a $\Bbbk$-*algebra* is a $\Bbbk$-vector space endowed with a multiplication that is compatible with the vector space structure. We say that an algebra is *finite-dimensional* if its dimension as a $\Bbbk$-vector space is finite. Classical examples of such structures are matrix algebras. Consider, for instance, the set of 2 by 2 lower-diagonal matrices over a field $\Bbbk$: $\Lambda_{A_2} = \left\{ \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \mid a, b, c \in \Bbbk \right\}$, then Λ_{A_2} is a finite-dimensional $\Bbbk$-vector space (of dimension 3) equipped with a multiplication (standard matrix multiplication) which is compatible with addition and scalar multiplication, thereby defining a $\Bbbk$-algebra. The algebra Λ_{A_2} can be described via the combinatorics of a finite directed graph, also known as *quiver*. Indeed, let A_2 be the quiver with two *vertices* and one *arrow*

$$1 \xrightarrow[\alpha]{} 2 \,.$$

To A_2 we can associate the vector space $\Bbbk A_2 = \Bbbk e_1 \oplus \Bbbk e_2 \oplus \Bbbk\alpha$, where e_i is the path corresponding to vertex i for $i = 1, 2$, and α is the path that follows the only arrow going from 2 to 1. By equipping $\Bbbk A_2$ with the product induced by path concatenation, we endow $\Bbbk A_2$ with the structure of a $\Bbbk$-algebra. The algebra $\Bbbk A_2$ is an example of a *path algebra* over a quiver. Moreover, the map given by $e_1 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\alpha \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ induces an algebra isomorphism between Λ_{A_2} and $\Bbbk A_2$. When $\Bbbk$ is an algebraically closed field, any *basic* finite-dimensional $\Bbbk$-algebra Λ is isomorphic to the quotient of a path algebra $\Bbbk Q_\Lambda$ by an ideal I_Λ of *admissible relations*. This fact was first observed by P. Gabriel in the early 70's [Gab72], and since quivers have become an essential tool in the study of finite-dimensional algebras.

A (right) *module* over a finite-dimensional $\Bbbk$ algebra Λ is a $\Bbbk$-vector space endowed with a (right) Λ action. The collection of all Λ-modules together with the Λ-linear morphisms between them is an example of a *abelian category*, denoted by $\operatorname{mod}\Lambda$. When Λ is the path algebra $\Bbbk Q/I$ of a quiver with relations (Q, I), the category $\operatorname{mod}\Lambda$ is equivalent to the category of *representations* $\operatorname{rep}(Q^{op}, I^{op})$ of the opposite quiver with relations (Q^{op}, I^{op}). A *representation* of a quiver with relations (Q, I) is the assignation of a finite-dimensional vector space to each vertex and a linear map to each arrow, such that the composition of linear maps corresponding to the relations in I vanishes.

Semistability and projective presentations

The category $\operatorname{mod}\Lambda$ satisfies the very interesting property that any Λ-module can be decomposed as an unique (up to isomorphism) direct sum of finitely many *indecomposable* modules, that is, modules that cannot be decomposed in a direct sum of two proper submodules. One of the fundamental theorems in the representation theory of finit-dimensional algebras is the *tame-wild dichotomy* [Dro06, DG92], which implies that the classification of the indecomposable modules in $\operatorname{mod}\Lambda$ is hopeless and cannot be achieved solely using enumerative tools. When traditional counting methods become insufficient, mathematicians often turn to analyzing the geometry of the problem. In representation theory, this approach led to the introduction of the *moduli space of representations* (see [Kin94, Rei08]), an algebraic variety whose points parametrize portions of the category of representations. Given a quiver with relations (Q, I), the set of representations of fixed *dimension vector* forms a variety equipped with a algebraic group action whose orbits consists of isomorphic representations. In this setting, the problem of classifying all representations of (Q, I) can be thought as the study of the sets of orbits for each dimension vector. The sets of all orbits are rarely algebraic varieties, but certain subsets of orbits can be considered. On how to choose which orbits to pick, D. Mumford provided an answer for quasi-projective varieties under the action of reductive algebraic groups: take orbits of points that are *χ_θ-semistable* (Definition 2.2, [MFK94]). Moduli spaces of representations are the varieties of orbits of χ_θ-semistable representations (up to an equivalence relation). In representation theory, these χ_θ can be encoded with integer vectors θ, and are referred to as *stability conditions.*

$$\begin{tikzcd}
 & & & 4 \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ru] \arrow[rd] & \\
 & & & 5
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & \mathbb{C} \\
\mathbb{C} \arrow[r, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"] & \mathbb{C}^2 \arrow[r, "{(1,-1)}"] & \mathbb{C} \arrow[ru, "1"] \arrow[rd, "0"'] & \\
 & & & 0
\end{tikzcd}$$

Figure 2: Example of a quiver and a representation. The dimension vector of this representation is $(1, 2, 1, 1, 0)$.

Since its introduction in the 1990s, semistability has occupied a central role in representation theory. One result showcasing its importance is the one obtained by C. Ingalls and H. Thomas in [IT09] . They showed that *left finite* subcategories of semistable representations of hereditary quivers Q are in correspondence with a diverse range of objects in the category of representations, namely *functorially finite torsion pairs* and *support τ-tilting pairs*, as well as related combinatorial objects associated to Q like *non-crossing partitions* (for *Dynkin* quivers) and clusters in the *cluster algebra* given by Q [IT09, Theorem 1.1]. The relation between these different aspects has continued to expand, and is still a very active line of research in representation theory [BMR$^+$06, AIR14, MŠ17, Yur18, BST19, DIJ19, BH23, AP22, BH23, BDM$^+$20, STW16].

Geometrically, a representation is θ-semistable if there exists a regular function, known as *θ-semi-invariant* or semi-invariant of *weight* θ, that does not vanish when evaluated on the point associated to the representation. A key ingredient in the construction of said regular functions is the use *projective presentations* [AB69, AR73, Pau08, PZ23]. A projective presentation of a given a Λ-module M is a map $\begin{array}{c} X_M^{-1} \\ \downarrow{\scriptstyle x_M} \\ X_M^0 \end{array}$ between *projective* modules X_M^{-1} and X_M^0 such that the cokernel of x_M is isomorphic to M. When Λ is a finite-dimensional algebra, all modules in $\operatorname{mod}\Lambda$ admit a projective presentation. Given two projective presentations $\begin{array}{c} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^0 \end{array}$ and $\begin{array}{c} Y^{-1} \\ \downarrow{\scriptstyle y} \\ Y^0 \end{array}$, a morphism between them is given by a commutative square

$$\begin{tikzcd}
X^{-1} \arrow[r, "f^{-1}"] \arrow[d, "x"] & Y^{-1} \arrow[d, "y"] \\
X^0 \arrow[r, "f^0"] & Y^0.
\end{tikzcd}$$

The collection of all projective presentations, also known as *2-term complexes*, along with their morphisms up to *homotopy*, forms an *extriangulated category* denoted as $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ (Definition 1.8, [NP19]). Similar to representations and their dimension vectors, we can categorize projective presentations using an integer vector known as *g-vector*, which represents the class of a projective presentation in the Grothendieck group $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$. Given a quiver and a stability condition θ, all θ-semi-invariants are generated by certain polynomials $s(X,-)$, known as *determinantal semi-invariants* (Definition 2.6), first introduced by A. Schofield in [Sch91]. These polynomials are constructed using projective presentations X whose g-vectors are a positive multiple of θ [DW00, SVdB01, Dom02]. Thus, to verify if a representation M is θ-semistable, it is sufficient to find a projective presentation X whose g-vector is a positive multiple of θ such that $s(X,M)$ is non-zero. Using the Hilbert-Mumford criterion, A. King showed that being θ-semistable possess a straightforward algebraic interpretation [Kin94]. He showed that a representation M is θ-semistable if and only if a particular numerical relationship between its dimension vector and the stability condition θ is met (Proposition 2.4) : θ must be orthogonal to the dimension vector of M, and for any non-trivial subrepresentation $N \subset M$, the scalar product between θ and the dimension vector of N should be less than or equal to zero.

Tropical duality and semistability

Important examples of stability conditions are those given by the g-vectors of 2-term *silting* complexes (Definition 3.27). Their associated categories of semistable modules are precisely those shown to be in correspondence with torsion classes and support τ-tilting pairs. They are also known to play a crucial role in the categorification of *cluster algebras* [FZ02]. Indeed, when Λ is the Jacobi algebra of a quiver with non-degenerate potential (Q,W), *reachable* 2-term silting complexes are in bijection with *clusters* of the cluster algebra associated to Q [CIKLFP13]. In particular, the g-vectors of reachable silting complexes coincide with integer vectors that determine the variables in the associated cluster, also know as g-vectors [4] [FZ07, Proposition 2.10].

For every cluster, there exists another important set of integer vectors known as *c-vectors*, which correspond to exponents of the *coefficients* related to the cluster variables. Both the set of g-vectors and the set of c-vectors associated with a

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[4] The term *g-vector* for the classes of 2-term complexes in the Grothendieck group $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$ is derived from cluster theory.

given cluster form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ and are, in fact, dual to each other [NZ12]. This duality, known as *tropical duality*, extends beyond a mere numerical fact: it can be interpreted as a reflection of the interactions between the categories $\operatorname{mod}\Lambda$ and $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ when Λ is a Jacobi algebra. For instance, c-vectors are known to correspond to dimension vectors of certain modules known as *bricks* [Nag13, NC13] (Definition 4.23). In [Tre19], H. Treffinger showed that the bricks whose dimension vectors correspond to the c-vectors associated to the cluster of a 2-term silting complex, could be obtained by considering subcategories of semistable representations for stability conditions given by the associated g-vectors [Tre19, Theorem 3.5]. In particular, the duality between the basis of c-vectors and g-vectors of a given 2-term silting complex follows from the numerical conditions defining semistability. In short, g-vectors turn out to be stability conditions for c-vectors. Additionally, in [IOTW09], K. Igusa, K. Orr, G. Todorov, and J. Weyman showed that determinantal semi-invariants $s(-,M)$ for M with dimension vector δ orthogonal to a given θ, define δ-*virtual* semi-invariants for projective presentations with g-vector θ (Section 2.2.1). Furthermore, in a subsequent article [IOTW15], they established that when Λ is a finite-dimensional hereditary algebra, the c-vectors associated with a 2-term silting complex are weights of determinantal semi-invariants (up to a sign). This raises the question: could c-vectors be thought of as stability conditions for g-vectors?

Thesis Outline and Contributions

In this thesis, we study the category of 2-term complexes $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ in the context of semistability and τ-tiling theory. Our aim is to explore the possibility of defining a notion of semistability for projective presentations that reflects both geometric and homological properties akin to those for representations.

The thesis is organized as follows. In Chapter 1, we revisit the definition of an extriangulated category and present the homotopy category of 2-term complexes of projective modules $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ over a finite-dimensional algebra Λ in this context. Throughout, we will refer to $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ as *the category of projective presentations.* We recall classical properties of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, and provide elementary proofs for most of them. In particular, we recall that $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is a 0-*Auslander extriangulated category* in the sense of [GNP23]. In this first chapter, we introduce the notion which is central to this manuscript: that of *thick subcategory* (Definition 1.16) of an extriangulated category, as defined in [NOS22]. Chapter 2 is devoted to the geometric notion of semistability and that of determinantal semi-inivariant. We show that the existence of a non-vanishing virtual semi-invariant of weight δ over varieties of projective presentations implies a numerical condition between the weight δ and g-vectors of 2-term complexes, which is dual to that discovered by A. King for semistable modules; we also exhibit a case where the converse does not hold. We provide as well an example showcasing that, unlike the subcategories of modules that are semistable in the sense of A. King, the subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ which are *numerically semistable* for certain weight δ, do not define well-behaved subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. We propose instead another notion, that of M-*semistability* for $M \in \operatorname{mod}\Lambda$ (Definition 2.15 and 3.66), which is defined by determinantal semi-invariants.

Subcategories of M-semistable modules are an example of thick subcategories of

$\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. In Chapter 3, after revisiting τ-tilting theory following T. Adachi, O. Iyama, and I. Reiten, as well as the Ingalls-Thomas' correspondences, we establish that thick subcategories with *enough injectives* are in correspondence with *cotorsion pairs* and 2-term presilting complexes. We show that these new bijections mirror those between wide subcategories, torsion pairs, and support τ-tilting pairs first discovered by C. Ingalls and H. Thomas. In Chapter 4 we study the case when Λ is g-finite, that is, when Λ possesses finitely many isomorphism classes of 2-term silting complexes. L. Demonet, O. Iyama and G. Jasso showed that an algebra is g-finite if and only if all torsion classes are functorially finite; or equivalently, if and only if there is a finite number of wide subcategories [5]. We show that a dual statement holds for $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, namely, Λ is g-finite if and only if all cotorsion pairs are complete, and if and only if there a finitely many thick subcategories. Finally, we show that when Λ is a g-finite algebra, all thick subcategories have enough injectives.

The results of Chapters 2 and 3 have been submitted for publication and are available at [Gar23]. Those of Chapter 4 were presented at the Oberwolfach workshop *Cluster Algebras and Its Applications* in January 2024.

Numerical semistability for projective presentations

Let Λ be a finite-dimensional basic $\mathbb{C}$-algebra, and let $n = |\Lambda|$ be the number of its non-isomorphic indecomposable summands. Consider $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ the full subcategory of 2-term complex of the homotopy category of bounded complexes of projective Λ-modules $\mathcal{K}^b(\operatorname{proj}\Lambda)$. For any $X = \begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^0 \end{smallmatrix}$ we denote by $[X]$ the class of X in the Grothendieck group $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$, which is equal to the the lattice $\bigoplus_{i=1}^n \mathbb{Z}[P_i]$, where the P_i are the n-indecomposable direct summands of Λ. The class $[X]$ can be identified with the integer vector

$$[X^0] - [X^{-1}] = \sum_{i=1}^n \theta_i^0 [P_i] - \sum_{j=1}^n \theta_j^{-1} [P_j]$$

where θ_i^{-1} and θ_i^0 satisfy that $X^{-1} \simeq \bigoplus_{i=1}^n P_i^{\oplus \theta_i^{-1}}$ and $X^0 \simeq \bigoplus_{i=1}^n P_i^{\oplus \theta_i^0}$. Recall that there exists a pairing $\langle -,- \rangle$ between $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$ and $K_0(\operatorname{mod}\Lambda)$ such that for any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $M \in \operatorname{mod}\Lambda$

$$\langle [X], [M] \rangle = \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^0, M)) - \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^{-1}, M)).$$

For any $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$, we consider the *virtual representation space* (Definition 2.16) as introduced in [IOTW09]:

$$R^{vir}(\theta) = \varinjlim_{\gamma \in \mathbb{Z}^n_{\geq 0}} \operatorname{Hom}_\Lambda(P^{\theta^-} \oplus P^\gamma, P^{\theta^+} \oplus P^\gamma)$$

where $\theta^- = -(\min(0,\theta_i))_{1 \leq i \leq n}$, $\theta^+ = (\max(0,\theta_i))_{1 \leq i \leq n}$ and $P^\eta = \bigoplus_{i=1}^n P^{\oplus \eta_i}$ for all $\eta \in \mathbb{Z}^n_{\geq 0}$. A *virtual semi-invariant* is an element of the ring

$$SI^{vir}(\theta) = \varprojlim_{\gamma \in \mathbb{Z}^n_{\geq 0}} SI(\operatorname{Hom}_\Lambda(P^{\theta^-} \oplus P^\gamma, P^{\theta^+} \oplus P^\gamma))^{\operatorname{Aut}_\Lambda(P^{\theta^-} \oplus P^\gamma) \times \operatorname{Aut}_\Lambda(P^{\theta^+} \oplus P^\gamma)},$$

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[5] Their bijection is stated in terms of *bricks*, which correspond to simple objects of wide subcategories of $\operatorname{mod}\Lambda$.

where $SI(\mathrm{Hom}_\Lambda(P^{\theta^-}\oplus P^\gamma, P^{\theta^+}\oplus P^\gamma)^{\mathrm{Aut}_\Lambda(P^{\theta^-}\oplus P^\gamma)\times \mathrm{Aut}_\Lambda(P^{\theta^+}\oplus P^\gamma)}$ denotes the ring of semi-invariants of the $\Bbbk$-vector space $\mathrm{Hom}_\Lambda(P^{\theta^-}\oplus P^\gamma, P^{\theta^+}\oplus P^\gamma)$ under the action of the algebraic group $\mathrm{Aut}_\Lambda(P^{\theta^-}\oplus P^\gamma)\times \mathrm{Aut}_\Lambda(P^{\theta^+}\oplus P^\gamma)$.

Proposition ([IOTW09, Proposition 5.13])**.** *Let $X^{-1}, X^0 \in \operatorname{proj}\Lambda$ and $M \in \operatorname{mod}\Lambda$, such that $\langle [X^0]-[X^{-1}], [M]\rangle = 0$. Then the regular function*

$$s(x, M) = \det\left(\mathrm{Hom}_\Lambda(X^0, M) \xrightarrow{\mathrm{Hom}_\Lambda(x,M)} \mathrm{Hom}_\Lambda(X^{-1}, M)\right)$$

is a $\mathrm{Aut}_\Lambda(X^{-1})\times\mathrm{Aut}_\Lambda(X^0)$-semi-invariant over $Hom_\Lambda(X^{-1}, X^0)$ with weight $([M],[M]) \in \mathbb{Z}^{2n}_{\geq 0}$. Moreover, it defines a virtual semi-invariant in $SI^{vir}(\theta)$ for $\theta = [X^0]-[X^{-1}]$.

The regular functions $s(-,-)$ are known as *determinental semi-invariants*. They were first introduced by A. Schofield in [Sch91], and were subsequently shown to generate the semi-invariant rings of representation varieties [SVdB01]. In particular, to check if a module is θ-semistable it suffices to find $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $m \in \mathbb{Z}_{\geq 0}$ such that $[X] = m\theta$ and $s(X, M) \neq 0$. This inspires the following definition.

Definition (2.15, M-stability). Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $M \in \operatorname{mod}\Lambda$. We say that X is *M-semistable* if

i) $\langle [X], [M]\rangle = 0$;

ii) $s(X, M) \neq 0$.

As we have mentioned in the previous section, when studying semistability for modules one could forgo of geometry completely and decide whether a module is χ_θ-semistable for a given $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$ by checking a simple numerical condition:

Proposition ([Kin94, Proposition 3.1])**.** *Let $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$, then $M \in \operatorname{mod}\Lambda$ is (geometrically) χ_θ-semistable if and only if*

i) $\langle \theta, [M]\rangle = 0$;

ii) For every submodule $N \subset M$, $\langle \theta, [N]\rangle \leq 0$.

Moreover, the subcategory $\mathscr{W}_\theta$ of θ-semistable modules is an exact abelian subcategory of $\operatorname{mod}\Lambda$.

We say that a morphism $f : Y \to X$ in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is an *inflation* if there exists a triangle $Y \xrightarrow{f} X \to Z \dashrightarrow Y[1]$ such that $Z \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. The previous proposition leads to the following definition.

Definition (2.21, Numerical semistability). Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $d \in \mathcal{K}_0(\operatorname{mod}\Lambda)$. We say that X is *numerically d-semistable* if

i) $\langle [X], d\rangle = 0$,

ii) For every inflation $Y \rightarrowtail X$ we have $\langle [Y], d\rangle \geq 0$.

The principal result of Chapter 2 establishes the relationship between numerical semistability and M-semistability.

Proposition (2.22). *Let* $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *and* $M \in \operatorname{mod}\Lambda$ *with corresponding class* $[M]$ *in* $\mathcal{K}_0(\operatorname{mod}\Lambda)$*. If* X *is* M*-semistable, then* X *is numerically* $[M]$*-semistable.*

We provide and example were the converse of Proposition 2.22 does not hold. Indeed, in Example 2.24 we present a non-hereditary algebra Λ for which we can find $\delta \in K_0(\operatorname{mod}\Lambda)$ and $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfying that X is numerically δ-semistable, but such that there is no semi-invariant f of weight δ (in particular no determinantal semi-invariant), over $\operatorname{Hom}_\Lambda(X^{-1} \oplus P^\gamma, X^0 \oplus P^\gamma)$ for any $\gamma \in \mathbb{Z}^n_{\geq 0}$ such that $f(X) \neq 0$. Lastly, we show that categories of δ-numerical semistable projective presentations are not closed under extensions in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ in general, highlighting the limitations of a straightforward numerical notion of semistability for projective presentations dual to King's condition for modules.

Ingalls-Thomas' correspondences for projective presentations

Given a stability condition $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$, the full subcategory $\mathscr{W}_\theta$ of modules that are θ-semistable forms an exact abelian subcategory $\operatorname{mod}\Lambda$, also referred to as *wide subcategory.* Besides their geometric significance, semistable subcategories of modules are relevant in representation theory due to their ties to *τ-tilting theory.* This connection was first brought to light by C. Ingalls and H. Thomas in [IT09], who provided correspondences between the following sets:

1) the set sτ-tilt Λ of isomorphism classes of basic support τ-tilting pairs[6] (Definition 3.2),

2) the set f-tors Λ of functorially finite torsion classes of $\operatorname{mod}\Lambda$ (Definition 3.3),

3) the set l-wide Λ of left finite wide subcategories $\operatorname{mod}\Lambda$ (Definition 3.20),

4) the set of left finite semistable subcategories of $\operatorname{mod}\Lambda$,

when Λ is an *hereditary* finite-dimensional algebra. Bijections between 1), 2) and 3) for any algebra Λ were shown by F. Marks and J. Šťovíček in [MŠ17]. The generalization of the bijection between 3) and 4) was independently obtained by T. Yurikusa in [Yur18] and T. Brüstle, D. Smith and H. Treffinger in [BST19]. In fact, all left finite wide subcategories can be realized as subcategories of modules which satisfy that the determinantal semi-invariant $s(X,-)$ does not vanish for certain 2-term *presilting* complex $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$:

Theorem (2.10, [Yur18, BST19]**).** *For any left finite wide subcategory* $\mathcal{W}$*, there exists a* 2*-term presilting complex* U *of* g*-vector* θ *such that*

$$\mathcal{W} = \mathscr{W}_\theta = \mathscr{W}(U) = \{M \in \operatorname{mod}\Lambda \mid \langle\theta, [M]\rangle = 0 \text{ and } s(U, M) \neq 0\}.$$

Determinantal semi-invariants also give rise to subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with interesting properties :

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[6] In their original article, this correspondence is expressed in terms of cluster tilting objects, who were shown to be in bijection with support τ-tilting pairs in [AIR14].

Proposition (3.70). *Let $M \in \operatorname{mod}\Lambda$, then the full subcategory*

$$\mathscr{T}(M) = \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid s(X, M) \neq 0\}$$

is thick, that is, it is additive and satisfies that for all triangles $X \to Y \to Z \dashrightarrow X[1]$ with $X, Y, Z \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, if two of the objects appearing in the triangle lie in $\mathscr{T}(M)$, then the third does as well. In other words, $\mathscr{T}(M)$ is closed under extensions, cones and cocones in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Inspired by the previous proposition, we introduce new bijections between thick subcategories and known classes of objects in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, namely *complete cotorsion pairs* (Definition 3.40) and 2-term silting complexes. These new correspondences mirror those between left finite wide subcategories, functorially finite torsion classes and support τ-tilting pairs.

Theorem (3.51). *Let Λ be a finite-dimensional $\Bbbk$-algebra. There exist well-defined maps*

$$\begin{tikzcd} \operatorname{cotor}\Lambda \arrow[r, shift left, "\beta"] & \operatorname{thick}\Lambda \arrow[l, shift left, "I"] \end{tikzcd}$$

between the set c-cotor Λ *of cotorsion pairs and the set* thick Λ *of thick subcategories in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. They are given by*

$$\beta((\mathcal{X}, \mathcal{Y})) = \{X \in \mathcal{X} \mid \forall\ X \to X' \to X'' \dashrightarrow X[1] \text{ such that } X' \in \mathcal{X},\ \text{then } X'' \in \mathcal{X}\}$$

$$I(\mathcal{T}) = \left({}^{\perp_1}(\mathcal{T}^{\perp_1}), \mathcal{T}^{\perp_1}\right),$$

for all $(\mathcal{X}, \mathcal{Y}) \in \operatorname{cotor}\Lambda$ and $\mathcal{T} \in \operatorname{thick}\Lambda$. Furthermore, when restricted to the set c-cotor Λ *of complete cotorsion pairs and the set* inj-thick Λ *of thick subcategories with enough injectives, β and I are inverse of each other.*

Theorem (3.72). *Let Λ be a finite-dimensional $\Bbbk$-algebra. There exist well defined maps*

$$\begin{tikzcd} \operatorname{wide}\Lambda \arrow[r, shift left, "\mathscr{T}"] & \operatorname{thick}\Lambda \arrow[l, shift left, "\mathscr{W}"] \end{tikzcd}$$

where for all $\mathcal{W} \in \operatorname{wide}\Lambda$ and $\mathcal{T} \in \operatorname{thick}\Lambda$

$$\mathscr{T}(\mathcal{W}) = \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid X \text{ is } M\text{-semistable } \forall\ M \in \mathcal{W}\}$$
$$\mathscr{W}(\mathcal{T}) = \{M \in \operatorname{mod}\Lambda \mid X \text{ is } M\text{-semistable } \forall\ X \in \mathcal{T}\}.$$

Moreover, the map $\mathscr{W}$ is a bijection between the set of thick subcategories with enough injectives and the set of left finite wide subcategories.

Corollary (3.33). *There are explicit bijections between*

1) Isomorphism classes of basic 2-term silting complexes in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

2) Complete cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

3) Thick subcategories in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with enough injectives.

These bijections are compatible with those between support τ-tilting pairs, functorially finite torsion classes and left finite wide subcategories; in other words, the following diagram is commutative:

$$\begin{tikzcd}
& & \text{2-silt}\,\Lambda \arrow[dl, "\Psi"', "{[AT22]}"] \arrow[dr, "\mathrm{thick}(U_\rho)"] \arrow[dd, "H^0"', "{[AIR14]}"] & & \\
\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) & \text{c-cotor}\,\Lambda \arrow[rr, "\beta"] \arrow[dd, "\Phi"', "{[PZ23]}"] & & \text{inj-thick}\,\Lambda \arrow[dd, "\mathscr{W}"] & \\
& & s\tau\text{-tilt}\,\Lambda \arrow[dl, "\mathrm{Fac}"', "{[AIR14]}"] \arrow[dr] & & \operatorname{mod}\Lambda \\
& \text{f-tors}\,\Lambda \arrow[rr, "\alpha", "{[M\check{S}17]}"'] & & \text{l-wide}\,\Lambda &
\end{tikzcd}$$

Thick subcategories for g-finite algebras

An algebra Λ is *g-finite* is it has finitely many isomorphism classes of basic support τ-tilting pairs (or equivalently if it has finitely many isomorphism classes of 2-term silting complexes). This notion was introduced and thoroughly studied by L. Demonet, O. Iyama and G. Jasso in [DIJ19], who show the following:

Theorem (4.2). *[DIJ19, Theorems 3.8 and 4.2] Let Λ be a finite-dimensional $\Bbbk$-algebra. The following are equivalent:*

1) Λ is g-finite.

2) There exist finitely many functorially finite torsion classes in $\operatorname{mod}\Lambda$.

3) All torsion classes in $\operatorname{mod}\Lambda$ *are functorially finite.*

4) There exist finitely many bricks in $\operatorname{mod}\Lambda$.

The following corollary follows from the definition of a left finite wide subcategory and the previous theorem.

Corollary (4.3). *Let Λ be a finite-dimensional $\Bbbk$-algebra. If Λ is g-finite, then all wide subcategories are left finite.*

In Chapter 4, we study the mirror counterparts of both Theorem 4.2 and Corollary 4.3 in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. An important tool used in the proofs of our results is the *reduction* of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with respect to a presilting complex U [GNP23]. Consider T_U the Bongartz completion of U, C_U the dg (*differential graded*) algebra $\mathcal{E}\mathrm{nd}_b(T_U)/\langle e_U\rangle$ where e_U is the idempotent associated to U, and $\mathrm{per}(C_U)$ the associated category of perfect complexes over C_U.

Lemma (4.43). *Let $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a 2-term presilting complex and let $\mathcal{U} = \operatorname{add}(U)$. Consider $\mathcal{Z}_{\mathcal{U}} = \{X \in \mathcal{K}^b(\operatorname{proj}\Lambda) \mid \operatorname{Hom}_b(X, U[i]) = 0 = \operatorname{Hom}_b(U[-i], X)\, \forall i > 0\}$ and $\rho : \mathcal{K}^b(\operatorname{proj}\Lambda) \to \mathcal{K}^b(\operatorname{proj}\Lambda)/\operatorname{thick}_b(\mathcal{U})$ the localization functor. Let $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a thick subcategory such that $U \in \mathcal{H}$. Then*

$$\frac{\mathcal{H} \cap \mathcal{Z}_{\mathcal{U}}}{[U]} \simeq \rho(\mathcal{H}).$$

In particular, $\rho(\mathcal{H})$ is thick inside the extriangulated category

$$\rho(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) \simeq \operatorname{per}^{[-1,0]}(C_U) = C_U * C_U[1].$$

Theorem 4.43 is one of the main tools used in the proof of the following theorem.

Theorem (4.51). *Let Λ be a g-finite, finite-dimensional $\Bbbk$-algebra. Let $\mathcal{H}$ be any thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, then there exists a presilting complex $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $\mathcal{H} = thick_{[-1,0]}(U)$.*

The main results of Chapter 4 give an analog of Theorem 4.2 and Corollary 4.3.

Theorem (4.4). *Let Λ be a finite-dimensional $\Bbbk$-algebra. The following are equivalent:*

1) Λ is g-finite.

2) There exist finitely many complete cotorsion classes in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

3) All cotorsion classes in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ are complete.

4) There exist finitely many thick subcategories in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

The equivalences among 1), 2), and 3) stem from the following result.

Theorem (4.44). *Let Λ be a finite-dimensional $\Bbbk$-algebra. Then the functor $H^0 : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \to \Lambda$ induces a bijection*

$$\begin{aligned} H^0 : \operatorname{cotor}\Lambda &\longrightarrow \operatorname{tors}\Lambda \\ (\mathcal{X},\mathcal{Y}) &\longmapsto H^0(\mathcal{Y}). \end{aligned}$$

We note that map between cotorsion pairs and torsion pairs in Theorem 4.44 was first introduced by D. Pauksztello and A. Zvonareva in [PZ23], who showed that it induced a bijection between complete cotorsion pairs and functorially finite torsion classes. We extended the bijection to all cotorsion and torsion pairs.

The last result in Chapter 4 and this manuscript is the analog of Corollary 4.3 and its dual:

Theorem (4.56). *Suppose Λ is g-finite. Then all thick subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ have enough injectives.*

Theorem (4.58). *Suppose Λ is g-finite. Then all thick subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ have enough projectives.*

Corollary (4.59). *Suppose Λ is g-finite. Then all thick subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ have enough projectives and enough injectives.*

CHAPTER 1

The category of projective presentations

Extriangulated categories were introduced by H. Nakaoka and Y. Palu in [NP19] in an aim to find a common framework for results occurring in both exact and triangulated categories. They discovered that it was sufficient to only remember the information given by *extension* functors. Follow-up work showed that phenomena such as the *localization* of exact and triangulated categories [NOS22], as well as *mutation* of extension-rigid objects [GNP23] could be recovered using this framework. One of the first examples of categories that are not exact nor triangulated is the *category of projective presentations*, or 2-term complexes of projective modules, which we denote by $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. In this short chapter, we recall the definition of extriangulated category and introduce the terminology and notation that will be used throughout this document. We also offer an overview of the properties satisfied by $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, including several well-known and classical ones. These properties will be referenced frequently throughout this text. We have opted to include elementary proofs for most of these properties.

1.1 Extriangulated categories

In this section, we fix an additive category $\mathcal{C}$ as well as an additive bifunctor $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$, where Ab is the category of abelian groups. Later on, $\mathbb{E}$ will have as codomain the set of $\Bbbk$-vector spaces for some field $\Bbbk$.

Definition 1.1. [NP19, Definition 2.1] For any $X, Z \in \mathcal{C}$, an element $\delta \in \mathbb{E}(Z, X)$ is called an $\mathbb{E}$-*extension.* We will refer to the element $0 \in \mathbb{E}(Z, X)$ as a *split* $\mathbb{E}$-extension. A *morphism* $(f, h) : \delta \to \delta'$ from $\delta \in \mathbb{E}(Z, X)$ to $\delta \in \mathbb{E}(Z', X')$ is a pair of morphisms $f \in \operatorname{Hom}_{\mathcal{C}}(X, X')$ and $h \in \operatorname{Hom}_{\mathcal{C}}(Z, Z')$ such that

$$\mathbb{E}(Z, f)(\delta) = \mathbb{E}(h, X')(\delta') \in \mathbb{E}(Z, X').$$

Remark 1.2. We will often refer to $\mathbb{E}$-extensions only as extensions when the the choice of bifunctor $\mathbb{E}$ is clear from the context.

Example 1.3. Let Λ be a finite-dimensional $\Bbbk$-algebra. Consider $\mathcal{C} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ the homotopy category of chain complexes of projective Λ-modules $X = (X_i, x_i)_{i\in\mathbb{Z}}$ such that $X_i = 0$ for all $i \neq 0, -1$. We denote by $X = \begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^0 \end{smallmatrix}$ such complexes. We will equip $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with the bifunctor $\mathbb{E}_{[-1,0]}(Z,X) = \operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(Z, X[1])$. In this context, for $X, X', Z, Z' \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, a morphism (f,h) from $\delta \in \mathbb{E}_{[-1,0]}(Z,X)$ to $\delta' \in \mathbb{E}_{[-1,0]}(Z',X')$ is equivalent to a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "\delta"] \arrow[d, "h"] & X[1] \arrow[d, "{f[1]}"] \\
Z' \arrow[r, "\delta'"] & X'[1]
\end{tikzcd}$$

up to homotopy.

Definition 1.4. [NP19, Definition 2.7] Let $X, Z \in \mathcal{C}$. We say that two sequences of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y' \xrightarrow{g'} Z$ are *equivalent* if there exists an isomorphism $h \in \operatorname{Hom}_{\mathcal{C}}(Y, Y')$ such that the diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[dd, "h"] & \\
X \arrow[ur, "f"] \arrow[dr, "f'"'] & & Z \\
 & Y' \arrow[ur, "g'"'] &
\end{tikzcd}$$

is commutative. We denote by $[X \xrightarrow{f} Y \xrightarrow{g} Z]$ the equivalence class of the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Definition 1.5. [NP19, Definition 2.9] A *realization* $\mathfrak{s}$ of the bifunctor $\mathbb{E}$ is a correspondence that associates to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(Z,X)$ an equivalence class $\mathfrak{s}(\delta) = [X \xrightarrow{f} Y \xrightarrow{g} Z]$ satisfying that for any morphism $(u,w) : \delta \to \delta'$ with $\delta' \in \mathbb{E}(Z',X')$ and $\mathfrak{s}(\delta') = [X' \xrightarrow{f'} Y' \xrightarrow{g'} Z']$, there exists $v \in \operatorname{Hom}_{\mathcal{C}}(Y,Y')$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[d, "w"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z'
\end{tikzcd}$$

is commutative. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to *realize* δ if $\mathfrak{s}(\delta) = [X \xrightarrow{f} Y \xrightarrow{g} Z]$. Similarly, (u,v,w) *realizes* (u,w) if the previous diagram commutes.

Definition 1.6. [NP19, DEfintion 2.10] A realization $\mathfrak{s}$ of $\mathbb{E}$ is *additive* if

i) For any $X, Z \in \mathcal{C}$, the realization of the split $\mathbb{E}$-extension $0 \in \mathbb{E}(Z,X)$ is given by

$$\mathfrak{s}(0) = [X \xrightarrow{\binom{1_X}{0}} X \oplus Z \xrightarrow{(0\ 1_Z)} Z]$$

ii) For any two extensions $\delta \in \mathbb{E}(Z,X)$ and $\delta' \in \mathbb{E}(Z',X')$, the extension $\delta \oplus \delta' \in \mathbb{E}(Z \oplus Z', X \oplus X')$ is realized by

$$\mathfrak{s}(\delta \oplus \delta') = [X \oplus X' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} Y \oplus Y' \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}} Z \oplus Z']$$

where $\mathfrak{s}(\delta) = [X \xrightarrow{f} Y \xrightarrow{g} Z]$ and $\mathfrak{s}(\delta') = [X' \xrightarrow{f'} Y' \xrightarrow{g'} Z']$.

Example 1.7. Let $\mathcal{C} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Recall that in a triangulated category, such as $\mathcal{K}^b(\operatorname{proj}\Lambda)$, we denote by $\operatorname{Cone}(f)$ the cone of a morphism f. Let $X, Z \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $\delta \in \mathbb{E}_{[-1,0]}(Z, X)$, then the complex $\operatorname{Cone}(\delta)[-1]$ is concentrated in degrees -1 and 0. We let

$$\mathfrak{s}_{[-1,0]}(\delta) = [X \xrightarrow{\iota[-1]} \operatorname{Cone}(\delta)[-1] \xrightarrow{\pi[-1]} Z],$$

where ι and π are the morphisms appearing in the standard triangle $Z \xrightarrow{\delta} X[1] \xrightarrow{\iota} \operatorname{Cone}(\delta) \xrightarrow{\pi} Z[1]$. Recall that for any $\delta' \in \mathbb{E}_{[-1,0]}(X', Z')$, $f \in \operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(X, X')$ and $h \in \operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(X', Z')$ such that $f[1]\delta = \delta' h$, there exists $g[-1] : \operatorname{Cone}(\delta)[-1] \to \operatorname{Cone}(\delta)[-1]$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\iota[-1]"] \arrow[d, "f"] & \operatorname{Cone}(\delta)[-1] \arrow[r, "\pi[-1]"] \arrow[d, "g"] & Z \arrow[r, "\delta"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "\iota'[-1]"] & \operatorname{Cone}(\delta')[-1] \arrow[r, "\pi'[-1]"] & Z' \arrow[r, "\delta'"] & X'[1].
\end{tikzcd}$$

is commutative. Recall that for $0 \in \mathbb{E}_{[-1,0]}(Z, X)$, $\operatorname{Cone}(0)[-1] = X \oplus Z$ and that $\operatorname{Cone}(\delta \oplus \delta') \simeq \operatorname{Cone}(\delta) \oplus \operatorname{Cone}(\delta')$ for any δ and δ' as before. Hence $\mathfrak{s}_{[-1,0]}$ is an additive realization of $\mathbb{E}_{[-1,0]}$.

Definition 1.8. [NP19, Definition 2.12] A triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an *extriangulated category* if and only if

(ET1) $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$ is an additive bifunctor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) For any $\delta \in \mathbb{E}(Z, X)$ and $\delta' \in \mathbb{E}(Z', X')$ respectively realized by $[X \xrightarrow{f} Y \xrightarrow{g} Z]$ and $[X' \xrightarrow{f'} Y \xrightarrow{g'} Z]$, and any commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z',
\end{tikzcd}$$

there exists a morphism $(u, w) : \delta \to \delta'$ such that $g'v = wg$.

(ET3)^{op} Dual of (ET3).

(ET4) For any $\delta \in \mathbb{E}(Z', X)$ and $\delta' \in \mathbb{E}(X', Y)$ realized respectively by $[X \xrightarrow{f} Y \xrightarrow{f'} Z']$ and $[Y \xrightarrow{g} Z \xrightarrow{g'} X']$. Then there exists an object $Y' \in \mathcal{C}$ and a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "f'"] \arrow[d, "g"] & Z' \arrow[d, "d"] \\
X \arrow[r, "h"] & Z \arrow[r, "h'"] \arrow[d, "g'"] & Y' \arrow[d, "e"] \\
 & X' \arrow[r, equal] & X'
\end{tikzcd}$$

and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(Y', X)$ which satisfies that:

i) The sequence $Z' \xrightarrow{d} Y' \xrightarrow{e} X'$ realizes $\mathbb{E}(X', f')(\delta')$.

ii) $\mathbb{E}(d, X)(\delta'') = \delta$.

iii) The morphism $(f, e) : \delta'' \to \delta'$ is realized by $(f, 1_Z, e)$

(ET4)^{op} Dual of (ET4).

Remark 1.9. An extension-closed subcategory $\mathcal{K}$ of an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a subcategory satisfying that for any $\delta \in \mathbb{E}(Z, X)$ realized by $X \to Y \to Z \dashrightarrow$, if X and Z belong to $\mathcal{K}$, then Y does as well. Any extension-closed subcategory $\mathcal{K}$ is extriangulated when equipped with $\mathbb{E}' = \mathbb{E}\,|_{\mathcal{K}^{op}\times\mathcal{K}}$ and $\mathfrak{s}|_{\mathbb{E}'}$.

Example 1.10. Let $\mathcal{T}$ be a triangulated category with translation functor Σ. Then $\mathcal{T}$ is extriangulated when equipped with $\mathbb{E}_{\mathcal{T}}(Z, X) = \operatorname{Hom}_{\mathcal{T}}(Z, \Sigma X)$. The realization is given by $\Sigma^{-1}\operatorname{Cone}(-)$ as in Example 1.7. Axioms (ET3) and (ET3)^{op} follow from the third axiom of triangulated categories. Th octahedral axiom implies (ET4) and (ET4)^{op}.

Example 1.11. The category $(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda), \mathbb{E}_{[-1,0]}, \mathfrak{s}_{[-1,0]})$ is an extension-closed subcategory of the triangulated category $\mathcal{K}^b(\operatorname{proj}\Lambda)$, and thus it is extriangulated.

Proposition 1.12. *[NP19, Propositions 3.3 and 3.11] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $\delta \in \mathbb{E}(Z, X)$ be an $\mathbb{E}$-extension realized by $X \xrightarrow{f} Y \xrightarrow{g} Z$. Let $A \in \mathcal{C}$, then the following sequnces of abelian groups are exact:*

$$\begin{tikzcd}
\operatorname{Hom}(Z,A) \arrow[r, "\operatorname{Hom}(g,A)"] & \operatorname{Hom}(Y,A) \arrow[r, "\operatorname{Hom}(f,A)"] & \operatorname{Hom}(X,A) \arrow[dll, rounded corners, "\mathbb{E}(Z,-)(\delta)"] \\
\mathbb{E}(Z,A) \arrow[r, "\mathbb{E}(g,A)"] & \mathbb{E}(Y,A) \arrow[r, "\mathbb{E}(f,A)"] & \mathbb{E}(X,A)
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Hom}(A,X) \arrow[r, "\operatorname{Hom}(A,f)"] & \operatorname{Hom}(A,Y) \arrow[r, "\operatorname{Hom}(A,g)"] & \operatorname{Hom}(A,Z) \arrow[dll, rounded corners, "\mathbb{E}(-,X)(\delta)"] \\
\mathbb{E}(A,X) \arrow[r, "\mathbb{E}(A,f)"] & \mathbb{E}(A,Y) \arrow[r, "\mathbb{E}(A,g)"] & \mathbb{E}(A,Z)
\end{tikzcd}$$

Corollary 1.13. *[NP19, Corollaries 3.5 and 3.6] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Let $\delta \in \mathbb{E}(Z, X)$, $\delta' \in \mathbb{E}(Z', X')$ and let $(u, w) : \delta \to \delta'$ be a morphism of $\mathbb{E}$-extensions realized by*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "v"] & Z \arrow[d, "w"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z'.
\end{tikzcd}$$

Then the following hold:

i) If u and w are isomorphisms, then so is v.

ii) If u and v are isomorphisms, then so is w.

iii) If v and w are isomorphisms, then so is u.

Moreover, $\delta = 0$ if and only if f is a section if and only if g is a retraction.

Proof. The statements i), ii), and iii) follow from Proposition 1.12 by applying the five lemma to the appropriate commutative diagram. For the last property, remark that if δ splits, that is $\mathfrak{s}(\delta) = [X \xrightarrow{\binom{1_X}{0}} X \oplus Z \xrightarrow{(0\ 1_Z)} Z]$, then clearly f is a section and g is a retraction. Now suppose that f is a section, then there exists $f' \in \operatorname{Hom}_{\mathcal{C}}(Y, X)$ such that $f'f = 1_X$. Since the sequence

$$\operatorname{Hom}(Z,X) \xrightarrow{-\circ g} \operatorname{Hom}(Y,X) \xrightarrow{-\circ f} \operatorname{Hom}(X,X) \xrightarrow{\mathbb{E}(Z,-)(\delta)} \mathbb{E}(Z,X)$$

is exact, we deduce that $0 = (\mathbb{E}(X, - \circ f)(\delta))\,(f') = \mathbb{E}(X, f'f)(\delta) = \mathbb{E}(X, 1_X)(\delta) = \delta$, thus δ splits. □

1.1.1 Notation and Terminology

Concluding this section, we introduce the notation and terminology that will be used throughout this manuscript. Consider an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

i) A *conflation* will be any sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ that realizes an $\mathbb{E}$-extension $\delta \in \mathbb{E}(Z, X)$. A conflation will be denoted either by $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$ or by $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ when δ is clear from the context. We write $Y \in X * Z$, whenever there exists a conflation as before with middle term Y.

ii) An *inflation* will be any morphism $X \xrightarrow{f} Y$ that appears as the first morphism in a conflation $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$. An inflation will be written as $X \overset{f}{\rightarrowtail} Y$. Note that (ET3) and Corollary 1.13 imply that Z and g can be defined from f up to isomorphism. We will denote by $\operatorname{Cone}(f)$ such an Z.

iii) A *deflation* will be any morphism $Y \xrightarrow{g} Z$ that appears as the second morphism in a conflation $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$. A deflation will be written as $Y \overset{g}{\twoheadrightarrow} Z$. Note that $(\text{ET3})^{op}$ and Corollary 1.13 imply that X and f can be defined from g up to isomorphism. We will denote by $\operatorname{Cocone}(g)$ such an X.

Definition 1.14. An object X in $\mathcal{C}$ is called *projective* if $\mathbb{E}(X, Y) = 0$ for any $Y \in \mathcal{C}$. The extriangulated category *has enough projectives* if for any $Y \in \mathcal{C}$ there exists a deflation $X \twoheadrightarrow Y$ with $X \in \mathcal{C}$ projective. Dually, X will be called *injective* if $\mathbb{E}(Y, X) = 0$ for all $Y \in \mathcal{C}$. We say that $\mathcal{C}$ *has enough injectives* if for every $Y \in \mathcal{C}$ there exists an inflation $Y \rightarrowtail X$ with $X \in \mathcal{C}$ injective.

For any subcategory $\mathcal{D}$ of an extriangulated category $\mathcal{C}$, we define

$$\begin{aligned} \mathcal{D}^{\perp_1} &= \{X \in \mathcal{C} \mid \mathbb{E}(\mathcal{D}, X) = 0\} \\ {}^{\perp_1}\mathcal{D} &= \{X \in \mathcal{C} \mid \mathbb{E}(X, \mathcal{D}) = 0\}. \end{aligned}$$

1.2 Thick subcategories

A central notion in this dissertation is that of a *thick subcategory*. This concept was initially introduced within the framework of extriangulated categories by H. Nakaoka, Y. Ogawa, A. Sakai in [NOS22, Definition 4.1] in order to characterize localizations of extriangulated categories.

Definition 1.15. Let $\mathcal{C}$ be a full subcategory of an extriangulated category $\mathcal{K}$.

i) We say that $\mathcal{C}$ is closed under extensions (see Remark 1.9) if for every conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{K}$ where $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$ as well.

ii) $\mathcal{C}$ is closed under cones if for every conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{K}$ where $X, Y \in \mathcal{C}$, then $Z \in \mathcal{C}$ as well.

iii) $\mathcal{C}$ is closed under cocones if for every conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{K}$ where $Y, Z \in \mathcal{C}$, then $X \in \mathcal{C}$ as well.

Definition 1.16. [NOS22, Definition 4.1] Let $\mathcal{K}$ be an extriangulated category. We say that a subcategory $\mathcal{C} \subset \mathcal{K}$ is *thick*, if it is full, closed under direct summands, extensions, cones and cocones. For all $\mathcal{C} \subset \mathcal{K}$, we denote by $\operatorname{thick}(\mathcal{C})$ the smallest thick subcategory that contains $\mathcal{C}$. We write $\operatorname{thick}\mathcal{K}$ for the set of thick subcategories of $\mathcal{K}$. When $\mathcal{K} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, we will write

$$\operatorname{thick}\Lambda := \operatorname{thick}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda).$$

Example 1.17. Let $\mathcal{D}$ be a triangulated category with shift functor Σ. Recall that a (triangulated) thick subcategory of $\mathcal{D}$ is a full, additive subcategory $\mathcal{C} \subset \mathcal{D}$ which is triangulated with respect to $\Sigma|_{\mathcal{C}}$. In particular it is thick with respect to the extriangulated structure of $\mathcal{D}$ induced by $\mathbb{E}_{\mathcal{D}}(-,-) = \operatorname{Hom}_{\mathcal{D}}(-, \Sigma-)$. Reciprocally, suppose that $\mathcal{C} \subset \mathcal{D}$ is thick with respect to the extriangulated structure given by $\mathbb{E}_{\mathcal{D}}$. For any $X \in \mathcal{C}$ we have a conflation

$$X \rightarrowtail 0 \twoheadrightarrow \Sigma X \overset{1_{\Sigma X}}{\dashrightarrow}$$

where $0 \in \mathcal{C}$ since $\mathcal{C}$ is additive. Given that $\mathcal{C}$ is closed under cones, we get that $\Sigma X \in \mathcal{C}$. Then, for any triangle $X \to Y \to Z \dashrightarrow \Sigma X$ with $X, Y \in \mathcal{C}$, the induced conflation $Y \rightarrowtail Z \twoheadrightarrow \Sigma X$ satisfies that $Y, \Sigma X \in \mathcal{C}$. Since $\mathcal{C}$ is closed under extensions, $Z \in \mathcal{C}$. We conclude that $\mathcal{C}$ is a full, additive subcategory which is triangulated when equipped with $\Sigma|_{\mathcal{C}}$.

Let $\mathcal{K} \subset \mathcal{D}$ be an extension closed, full, subcategory of a triangulated category $\mathcal{D}$. For a full subcategory $\mathcal{C} \subset \mathcal{K}$ we will denote by $\operatorname{thick}_{\mathcal{K}}(\mathcal{C}) \subset \mathcal{K}$ the smallest thick subcategory containing $\mathcal{C}$ in $\mathcal{K}$ when seen as an extriangulated category. We will write $\operatorname{thick}_{\mathcal{D}}(\mathcal{C})$ for the smallest triangulated thick subcategory in $\mathcal{D}$ containing $\mathcal{C}$. When $\mathcal{D} = \mathcal{K}^b(\operatorname{proj}\Lambda)$ and $\mathcal{K} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ for a finite-dimensional algebra Λ, we write

$$\operatorname{thick}_b(\mathcal{C}) := \operatorname{thick}_{\mathcal{D}}(\mathcal{C})$$
$$\operatorname{thick}_{[-1,0]}(\mathcal{C}) := \operatorname{thick}_{\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)}(\mathcal{C}).$$

1.3 The extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$

In this section, we revisit key properties of the category of projective presentations $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ of a finite-dimensional $\Bbbk$-algebra Λ, viewed as an extriangulated category, which will be frequently referenced throughout this text. As seen in the previous section, $(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda), \mathbb{E}_{[-1,0]}, \mathfrak{s}_{[-1,0]})$ is an extriangulated category since is an extension closed subcategory of the triangulated category $\mathcal{K}^b(\operatorname{proj}\Lambda)$. For $a \leq b \in \mathbb{Z}$, we denote by $\mathcal{K}^{[a,b]}(\operatorname{proj}\Lambda)$ the full subcategory of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ of complexes concentrated in degrees $[a, b]$. We will write

$$\operatorname{Hom}_b(A, B) = \operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(A, B)$$

for any $A, B \in \mathcal{K}^b(\operatorname{proj}\Lambda)$. Similar notation will be used for the sets of endomorphims and that of automorphisms. Let $n = |\Lambda|$ be the number of non-isomorphic indecomposable summands of Λ. We denote by P_i for $0 \leq 1 \leq n$ said indecomposable projective modules. To each P_i we associate the simple module S_i satisfying $S_i \simeq P_i / \operatorname{rad} P_i$. We let

$$K_0(\operatorname{mod}\Lambda) \cong K_0(\mathcal{D}^b(\operatorname{mod}\Lambda)) = \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$$

be the Grothendieck group of $\operatorname{mod}\Lambda$. Similarly, we consider

$$K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) \simeq K_0(\mathcal{K}^b(\operatorname{proj}\Lambda)) = \bigoplus_{i=1}^{n} \mathbb{Z}[P_i]$$

the Grothendieck group of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. The Euler form associated to these two groups is the paring given by

$$\langle -, - \rangle : K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) \times K_0(\operatorname{mod}\Lambda) \longrightarrow \mathbb{Z}$$
$$([P_j], [S_i]) \mapsto \langle [P_j], [S_i] \rangle = \begin{cases} \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(P_i, S_i)) = \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(S_i, S_i)) & i = j \\ 0 & i \neq j\ . \end{cases}$$

In particular, for every $M \in \operatorname{mod}\Lambda$ and $X = \begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^0 \end{smallmatrix} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, this pairing is given by

$$\langle [X], [M] \rangle = \langle [X^0] - [X^{-1}], [M] \rangle = \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^0, M)) - \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^{-1}, M)).$$

Definition 1.18. Let $M \in \operatorname{mod}\Lambda$. A *minimal projective presentation* of Λ is an exact sequence

$$X_M^{-1} \xrightarrow{x_M} X_M^0 \xrightarrow{\pi_M} M \to 0$$

where $X_M^{-1}, X_M^0 \in \operatorname{proj}\Lambda$ and such that

$$\pi_M : X_M^0 \to M$$
$$x_M : X_M^{-1} \to \operatorname{Ker}\pi_M$$

are projective covers of M and $\operatorname{Ker}\pi_M$ respectively. Since projective covers are unique up to isomorphisms, so are projective presentations. We will denote by X_M a choice of minimal projective presentation $\begin{smallmatrix} X_M^{-1} \\ \downarrow x_M \\ X_M^0 \end{smallmatrix}$ of M.

Proposition 1.19. *Let $M, N \in \operatorname{mod}\Lambda$, then*

$$X_{M\oplus N} \simeq X_N \oplus X_M$$

where $X_N \oplus X_M$ is given by

$$\begin{tikzcd} X_N^{-1} \oplus X_M^{-1} \arrow[d, "\begin{pmatrix} x_N & 0 \\ 0 & x_M \end{pmatrix}"] \\ X_N^0 \oplus X_M^0 \end{tikzcd}$$

Proof. It suffices to show that $X_N \oplus X_M$ is a minimal projective presentation for $N \oplus M$. We first show that $\left(X_N^0 \oplus X_M^0, \left(\begin{smallmatrix} \pi_N & 0 \\ 0 & \pi_M \end{smallmatrix}\right)\right)$ is a projective cover. Let $H \in \operatorname{mod}\Lambda$ and $P \xrightarrow{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)} X_N^0 \oplus X_M^0$ such that $H \xrightarrow{\left(\begin{smallmatrix} \pi_N x \\ \pi_M y \end{smallmatrix}\right)} X_N^{-1} \oplus X_M^{-1}$ is surjective. In particular $\pi_N x$ and $\pi_M y$ are surjective. Since (N, π_N) and (M, π_M) are projective covers, by [ASS06, Lemma 5.6] we get that both x and y are surjective and thus $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is surjective. Applying [ASS06, Lemma 5.6] again, we conclude that $\left(X_N^0 \oplus X_M^0, \left(\begin{smallmatrix} \pi_N & 0 \\ 0 & \pi_M \end{smallmatrix}\right)\right)$ is a projective cover. Since $\operatorname{Ker}\left(\begin{smallmatrix} x_N & 0 \\ 0 & x_M \end{smallmatrix}\right) = \operatorname{Ker}\pi_N \oplus \operatorname{Ker}\pi_M \subset N \oplus M$, the previous argument can be applied to $X_N^{-1} \oplus X_M^{-1} \xrightarrow{\left(\begin{smallmatrix} x_N & 0 \\ 0 & x_M \end{smallmatrix}\right)} \operatorname{Ker}\pi_N \oplus \operatorname{Ker}\pi_M$ to show that it is a projective cover. ☐

The following proposition is well know (see for instance [Bau04, Proposition 3.5]). We include a proof for the convenience of the reader.

Proposition 1.20. *The category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is a Hom-finite, additive and Krull-Schmidt category. Moreover, all its indecomposable objects are of the form $P[1]$ or X_M, where P in an indecomposable projective Λ-module and X_M is the minimal projective presentation of an indecomposable module $M \in \operatorname{mod}\Lambda$.*

Proof. We only provide a proof of the last statement, namely, the classification of all indecomposable objects within $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Consider $X = \begin{array}{c} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^0 \end{array}$ in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Let $M = H^0(X)$ and $\begin{array}{c} X_M^{-1} \\ \downarrow{\scriptstyle x_M} \\ X_M^0 \end{array}$ its minimal projective presentation. Since we have that $X_M^0 \xrightarrow{\pi_M} M$ is surjective and that X^0 is projective, there exists $x^0 : X^0 \to X_M^0$ such that $\pi_M x^0 = \pi$. Since (X_M^0, π_M) is a projective cover, x^0 is a split epimorphism. In particular $x^0((x^0)^{-1}(\operatorname{Ker}\pi_M)) = \operatorname{Ker}\pi_M$ and

$$\pi((x^0)^{-1}(\operatorname{Ker}\pi_M)) = (\pi_M x^0)((x^0)^{-1}(\operatorname{Ker}\pi_M)) = \pi_M(\operatorname{Ker}\pi_M) = 0,$$

that is $(x^0)^{-1}(\operatorname{Ker}\pi_M) \subset \operatorname{Ker}\pi = \operatorname{Im} f$. We get that $x^0 f : X^{-1} \to \operatorname{Ker}\pi_M$ is surjective. Since X^{-1} is projective and (X_M^{-1}, x_M) is a projective cover of $\operatorname{Ker}\pi_M$, we get a split epimorphism $x^{-1} : X^{-1} \to X_M^{-1}$ such that $x_M x^{-1} = x^0 f$. Up to applying isomorphisms, we get a commutative diagram

$$\begin{tikzcd}
X^{-1} \arrow[r, "\simeq"] \arrow[d, "f"] & X_M^{-1} \oplus \bar{X}^{-1} \arrow[r, "\begin{pmatrix} 1_{X_M^{-1}} & 0 \end{pmatrix}"] \arrow[d, "f'=\begin{pmatrix} x_M & 0 \\ y & z \end{pmatrix}"'] & X_M^{-1} \arrow[d, "x_M"] \\
X^0 \arrow[r, "\simeq"] \arrow[d, "\pi"] & X_M^0 \oplus \bar{X}^0 \arrow[r, "\begin{pmatrix} 1_{X_M^0} & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \pi_M & 0 \end{pmatrix}"'] & X_M^0 \arrow[d, "\pi_M"] \\
M \arrow[r, equal] & M \arrow[r, equal] & M.
\end{tikzcd}$$

We are going to show that z is surjective. Given that $\bar{X}^0 \subset \operatorname{Ker}\begin{pmatrix} \pi_M & 0 \end{pmatrix} = \operatorname{Im} f'$, we get that $\bar{X}^0 = y(\operatorname{Ker} x_M) + \operatorname{Im} z$. But (X_M^{-1}, x_M) is a projective cover, so $\operatorname{Ker} x_M \subset \operatorname{rad} X_M^{-1}$, which in turn implies that

$$\bar{X}^0 = y(\operatorname{rad} \bar{X}^{-1}) + \operatorname{Im} z \subset \operatorname{rad} \bar{X}^0 + \operatorname{Im} z = \operatorname{Im} z.$$

Since X_M^{-1} is projective and z is surjective, there exists $\alpha : X_M^{-1} \to \bar{X}_M^{-1}$ such that $y = z\alpha$. In particular, we have a commutative diagram

$$\begin{tikzcd}
X_M^{-1} \oplus \bar{X}^{-1} \arrow[r, "\begin{pmatrix} 1_{X_M^{-1}} & 0 \\ -\alpha & 1_{\bar{X}_M^{-1}} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} x_M & 0 \\ 0 & z \end{pmatrix}"] & X_M^{-1} \oplus \bar{X}^{-1} \arrow[d, "\begin{pmatrix} x_M & 0 \\ y & z \end{pmatrix}"] \\
X_M^0 \oplus \bar{X}^0 \arrow[r, equal] & X_M^0 \oplus \bar{X}^0.
\end{tikzcd}$$

Finally, since z is an epimorphism between projective modules, it must split. We get that $\bar{X}^{-1} \simeq \bar{X}^0 \oplus Q$ and that

$$\begin{tikzcd}
X^{-1} \arrow[r, "\simeq"] \arrow[d, "f"] & X_M^{-1} \oplus \bar{X}^0 \oplus Q \arrow[d, "\begin{pmatrix} x_M & 0 & 0 \\ 0 & 1_{\bar{X}^0} & 0 \end{pmatrix}"] \\
X^0 \arrow[r, "\simeq"] & X_M^0 \oplus \bar{X}^0.
\end{tikzcd}$$

That is, $X \simeq X_M \oplus Q[1]$ in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. By decomposing Q in indecomposable projective modules and by applying Proposition 1.19 to the decomposition of M in indecomposable summands in $\operatorname{mod}\Lambda$, we get the result. ☐

Proposition 1.21. *[Aus99, Chapter III] The additive functor*

$$\begin{aligned} H^0 : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) &\longrightarrow \operatorname{mod}\Lambda \\ X = \begin{array}{c} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^0 \end{array} &\longmapsto H^0(X) = \operatorname{Coker} x \end{aligned}$$

induces a equivalence of categories $H^0 : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)/[\operatorname{add}(\Lambda[1])] \xrightarrow{\simeq} \operatorname{mod}\Lambda$.

Proof. That the functor H^0 is essentially surjective and full follows from the existence of projective presentations and that any map between modules $M \xrightarrow{l} N$ lifts to a map $X_M \xrightarrow{l'} X_N$. Let $f \in \operatorname{Hom}_b(X_M, X_N)$ such that $H^0(f) = 0$. Hence, have a commutative diagram

$$\begin{tikzcd}
X_M^{-1} \arrow[r, "f^{-1}"] \arrow[d, "x_M"] & X_N^{-1} \arrow[d, "x_N"] \\
X_M^0 \arrow[r, "f^0"] \arrow[d, "\pi_M"] & X_N^0 \arrow[d, "\pi_N"] \\
M \arrow[r, "0"] & N,
\end{tikzcd}$$

which implies that f^0 factors through $\operatorname{Ker}\pi_N$. Since X_M^0 is projective and $x_N : X_N^{-1} \to \operatorname{Ker}\pi_N$ is surjective, there exists $k : X_M^0 \to X_N^{-1}$ such that $x_N k = f^0$. We get that f is homotopic to the map

$$\begin{tikzcd}
X_M^{-1} \arrow[r, equal] \arrow[d, "x_M"] & X_M^{-1} \arrow[r, "k\,x_M - f^{-1}"] \arrow[d] & X_N^{-1} \arrow[d, "x_N"] \\
X_M^0 \arrow[r] & 0 \arrow[r] & X_N^0
\end{tikzcd}$$

which factors through an object in $\operatorname{add}(\Lambda[1])$. This proves that when restricted to $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)/[\operatorname{add}(\Lambda[1])]$, H^0 is faithful. $\square$

1.3.1 $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is 0-Auslander

Definition 1.22. [GNP23, Section 3] We say that an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is 0-*Auslander* if it satisfies the following properties:

i) For every $X \in \mathcal{C}$, there exists a conflation $P' \rightarrowtail P \twoheadrightarrow X \dashrightarrow$ with P and P' projective objects.

ii) For every projective object P, there exists and inflation $P \rightarrowtail Q$ with Q projective-injective.

In other words $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is 0-Auslander if it has enough projectives, its *global dimension* is at most one and its *dominant dimension* is at least one [GNP23, Proposition 3.6]. A 0-Auslander extriangulated category is *reduced* if its only projective-injective object is 0 (up to isomorphism).

Remark 1.23. The previous definition suggests the existence of a dual notion, that is, an extriangulated category such that for every object $X \in \mathcal{C}$ there exists a conflation $X \rightarrowtail I \twoheadrightarrow I''$ where I, I'' are injective objects, and such that we can find an deflation $Q \twoheadrightarrow I$ for every injective I where Q is projective-injective. Remarkably, both notions are equivalent [GNP23, Proposition 3.6], rendering the definition of a 0-Auslander category self-dual.

Proposition 1.24. *[GNP21, Secation 3.3] Let Λ be a finite-dimensional $\Bbbk$-algebra. Then the category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is a reduced 0-Auslander extriangulated category.*

Proof. We will show that $\operatorname{proj}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \operatorname{add}(\Lambda)$ and that $\operatorname{inj}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \operatorname{add}(\Lambda)[1]$. First note that the complexes P and $P[1]$ are respectively projective and injective in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ for any $P \in \operatorname{proj}\Lambda$. Now suppose that $X = \begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^{0} \end{smallmatrix} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfies that $\mathbb{E}_{[-1,0]}(X,Y) = 0$ for every $Y \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Consider $1_{X^{-1}} \in \mathbb{E}_{[-1,0]}(X, X^{-1})$, since X is projective, $1_{X^{-1}}$ must be homotopic to 0, that is, there exists a commutative diagram

$$\begin{tikzcd} X^{-1} \arrow[r, "1_{X^{-1}}"] \arrow[d, "x"] & X^{-1} \arrow[d] \\ X^{0} \arrow[r] \arrow[ur, dashed, "k"] & 0, \end{tikzcd}$$

which implies that x is a section and $X \simeq \begin{smallmatrix} 0 \\ \downarrow \\ Q \end{smallmatrix}$ for some $Q \in \operatorname{proj}\Lambda$, hence $\operatorname{proj}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \operatorname{add}(\Lambda)$. Similarly, suppose that X is injective, then $\mathbb{E}_{[-1,0]}(Y,X) = 0$ for all $Y \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. If we consider now $1_{X^0} \in \mathbb{E}_{[-1,0]}(X^0[1], X)$, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X^{-1} \arrow[d, "-x"] \\ X^{0} \arrow[r, "1_{X^0}"'] \arrow[ur, dashed, "k"] & X^{0}. \end{tikzcd}$$

Then x is a retraction and $X \simeq \begin{smallmatrix} Q \\ \downarrow \\ 0 \end{smallmatrix}$ with $Q \in \operatorname{proj}\Lambda$. We conclude that $\operatorname{inj}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \operatorname{add}(\Lambda)[1]$ and thus the only projective-injective object in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is 0. Moreover, for every projective object P we have a conflation

$$P \rightarrowtail 0 \twoheadrightarrow P[1] \ ,$$

and for all $X = \begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^{0} \end{smallmatrix}$

$$X^{-1} \overset{x}{\rightarrowtail} X^{0} \twoheadrightarrow X \ ,$$

which gives de result. $\square$

Remark 1.25. The previous argument relies heavily on the structure of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ inherited from $\mathcal{K}^{b}(\operatorname{proj}\Lambda)$. It is also possible to prove that $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is a 0-Auslander extriangulated category "*from scratch*". Consider the category of morphisms between projective objects $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ and consider $\mathfrak{s}$ the set of sequences $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that for $i = -1, 0$, the sequence $X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i$ is exact. Together with $\mathfrak{s}$, $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ is an exact category (see for instance [Hap88]), and thus it is extriangulated. Its projective non-injective objects are given by the complexes of the form $\begin{smallmatrix} 0 \\ \downarrow \\ P \end{smallmatrix}$, its injective non-projectives are given by complexes $\begin{smallmatrix} Q \\ \downarrow \\ 0 \end{smallmatrix}$ and its projective-injective objects are of the form $\begin{array}{c} R \\ \| \\ R \end{array}$ with $P, Q, R \in \operatorname{proj}\Lambda$ [Bau04, Corollary 3.1 and 3.2]. Moreover, for any $\begin{array}{c} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^{0} \end{array}$ we can find an $\mathfrak{s}$-sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & X^{-1} \arrow[r, equal] \arrow[d, "\left(\begin{smallmatrix} 1_{X^{-1}} \\ 0 \end{smallmatrix}\right)"] & X^{-1} \arrow[d, "x"] \\
X^{-1} \arrow[r, "\left(\begin{smallmatrix} 1_{X^{-1}} \\ -x \end{smallmatrix}\right)"] & X^{-1} \oplus X^{0} \arrow[r, "(x\ 1_{X^{0}})"] & X^{0}.
\end{tikzcd}$$

In particular, $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ has enough projectives and for each projective object we have an $\mathfrak{s}$-sequence $\begin{tikzcd} 0 \arrow[r] \arrow[d] & P \arrow[r, equal] \arrow[d, equal] & P \arrow[d] \\ P \arrow[r, equal] & P \arrow[r] & 0 \end{tikzcd}$, whose middle term is projective-injective. We conclude that $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ is a 0-Auslander extriangulated category. The quotient of $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ by the ideal of morphisms that factor through a projective-injective object is precisely $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, since a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ factors through an object $\begin{array}{c} P \\ \| \\ P \end{array}$ if and only if it is homotopic to zero. That $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is 0-Auslander follows from the following proposition.

Proposition 1.26. *[GNP23, Proposition 3.2 and Corollary 3.3] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let J be an ideal generated by morphisms with injective domain and projective codomain. It $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is 0-Auslander, then so is C/J.*

Remark 1.27. The categories $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ appear frequently as quotients of more general 0-Auslander categories by the ideal of all morphisms from injective to projective objects [FGP^{+}23]. X. Chen showed that such quotients of 0-Auslander *algebraic* extriangulated categories are equivalent to a category $\mathcal{K}^{[-1,0]}(\mathcal{A})$ for certain additive category $\mathcal{A}$ [Che23]. Whether this statement holds in general is an open problem initially proposed in [FGP^{+}23].

Remark 1.28. Let $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ be a conflation in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and let x, y, z be the differentials of X, Y, Z respectively. Choose as well $h : Z^{-1} \to X^{0}$, a representative of the morphism $Z \dashrightarrow X[1]$ associated to the conflation (f, g). Then there is an isomorphism in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$

$$Y \cong \operatorname{Cocone}(Z^{-1} \dashrightarrow X[1])[-1] \cong \begin{array}{c} X^{-1} \oplus Z^{-1} \\ \big\downarrow{\scriptstyle \left(\begin{smallmatrix} x & h \\ 0 & z \end{smallmatrix}\right)} \\ X^{0} \oplus Z^{0} \end{array}.$$

If we choose a minimal representative y of the isomorphism class of Y, that is, such that it satisfies that $y \ncong \begin{pmatrix} y' & 0 \\ 0 & 1_{Q} \end{pmatrix}$ for all $0 \neq Q \in \operatorname{proj}\Lambda$, then there exists $P \in \operatorname{proj}\Lambda$ and a diagram

$$\begin{tikzcd}
Y^{-1} \oplus P \arrow[r, "\simeq"] \arrow[d, "\begin{pmatrix} y & 0 \\ 0 & 1_{P} \end{pmatrix}"] & X^{-1} \oplus Z^{-1} \arrow[d, "\begin{pmatrix} x & h \\ 0 & z \end{pmatrix}"] \\
Y^{0} \oplus P \arrow[r, "\simeq"] & X^{0} \oplus Z^{0}
\end{tikzcd}$$

that is commutative inside $\operatorname{mod}\Lambda$. That is, the obtained sequence $X \rightarrowtail Y \oplus \begin{array}{c} P \\ \shortparallel \\ P \end{array} \twoheadrightarrow Z$ is a conflation inside of $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$, whose image under the quotient of $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ by its projective-injective objects is precisely (f, g).

For any given extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, higher positive extension bifunctors $\mathbb{E}^i(-,-)$ $i \geq 1$ were defined in [GNP21]. For $\mathcal{K}^b(\operatorname{proj}\Lambda)$ and $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, positive extensions coincide with the bifunctors $\operatorname{Hom}_b(-,-[i])$ for $i \geq 1$. Hence, $\mathbb{E}^i_{[-1,0]}(X,Y) = \operatorname{Hom}_b(X, Y[i]) = 0$ for all $i \geq 2$ and $X, Y \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Proposition 1.29. *[GNP23, Proposition 2.1] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. The following properties are equivalent:*

i) The bifunctor $\mathbb{E}^2(-,-)$ is zero.

ii) The functor $\mathbb{E}(X,-)$ is right exact for any $X \in \mathcal{C}$.

iii) The functor $\mathbb{E}(-,Y)$ is right exact for any $Y \in \mathcal{C}$.

iv) Let $Y \overset{f}{\rightarrowtail} U \overset{f'}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$ and $Z \overset{d}{\rightarrowtail} T \overset{d'}{\twoheadrightarrow} X \overset{\delta'}{\dashrightarrow}$ be conflations in $\mathcal{C}$. Then there exists an object $V \in \mathcal{C}$ and a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, equal] & U \arrow[r, "f'"] \arrow[d, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "d"] & {} \\
Y \arrow[r, "h"] & V \arrow[r, "h'"] \arrow[d, "g'"] & T \arrow[r, dashed, "\eta"] \arrow[d, "d'"] & {} \\
& X \arrow[r, equal] \arrow[d, "\eta'"] & X \arrow[d, dashed, "\delta'"] & \\
& {} & {} &
\end{tikzcd}$$

where $U \overset{g}{\rightarrowtail} V \overset{g'}{\twoheadrightarrow} X \overset{\eta'}{\dashrightarrow}$ and $Y \overset{h}{\rightarrowtail} V \overset{h'}{\twoheadrightarrow} T \overset{\eta}{\dashrightarrow}$ are conflations such that the $\mathbb{E}$-extensions η and η' satisfy

(a) $\mathbb{E}(d, Y)(\eta) = \delta$,

(b) $\mathbb{E}(X, f')(\eta') = \delta'$,

(c) $\mathbb{E}(T, f)(\eta) = \mathbb{E}(d', Y)(\eta')$.

Moreover, if $\mathcal{C}$ has enough projectives, $i)$ is equivalent to every object $X \in \mathcal{C}$ admitting a conflation $P' \rightarrowtail P \twoheadrightarrow X$ where P and P' are projective in $\mathcal{C}$.

Definition 1.30. We say that an extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is *hereditary* if it satisfies one of the equivalent properties of Proposition 1.29.

Remark 1.31. If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a 0-Auslander extriangulated category, Proposition 1.29 says that it is hereditary. As we have seen, this is the case for $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Definition 1.32. An extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is said to satisfy *WIC*[1] if it satisfies the following properties:

---

[1] For weakly idempotent complete [BÏ0, Proposition 7.6].

i) If $h = fg$ is an inflation, then so is g.

ii) If $h = fg$ is an deflation, then so is f.

Other that being a Krull-Schmidt, Hom-finite, 0-Auslander extriangulated category, $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ also satisfies WIC. Indeed, consider $g : X \to Y$ and $f : Y \to Z$ such that $fg : X \to Z$ is a inflation. Then there exists $Z' \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and a conflation $X \overset{fg}{\rightarrowtail} Z \twoheadrightarrow Z'$. Using the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, we get a commutative diagram of triangles

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f"] & \operatorname{Cone}(g) \arrow[d] \\
X \arrow[r, "h"] & Z \arrow[r] \arrow[d] & Z' \arrow[d] \\
 & \operatorname{Cone}(f) \arrow[r, equal] & \operatorname{Cone}(f).
\end{tikzcd}$$

Since the last column fits into a triangle, $Z' \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $\operatorname{Cone}(f) \in \mathcal{K}^{[-2,0]}(\operatorname{proj}\Lambda)$, we deduce thar $\operatorname{Cone}(g)[1] \in \mathcal{K}^{[-2,0]}(\operatorname{proj}\Lambda)$. Hence $\operatorname{Cone}(g) \in \mathcal{K}^{[-2,0]}(\operatorname{proj}\Lambda) \cap \mathcal{K}^{[-1,1]}(\operatorname{proj}\Lambda) = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and g is an inflation. A similar argument shows that the dual statement holds.

1.3.2 Auslander-Reiten triangles and approximations in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$

Definition 1.33. Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory closed under isomorphisms of a category $\mathcal{C}$.

i) A *right $\mathcal{D}$-approximation* (D, f) of $X \in \mathcal{C}$ is a morphism $f : D \to X$ with $D \in \mathcal{D}$ that satisfies that, for every $f' : D' \to X$ with $D' \in \mathcal{D}$, there exists $k : D' \to D$ such that $f' = fk$. That is, $\operatorname{Hom}_{\mathcal{C}}(D', D) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(D',f)} \operatorname{Hom}_{\mathcal{C}}(D', X)$ is surjective. A right $\mathcal{D}$-approximation (D, f) is *minimal* if whenever $a \in \operatorname{End}_{\mathcal{C}}(D)$ satisfies $fa = f$, then a is an isomorphism.

ii) Dually, a *left $\mathcal{D}$-approximation* (D, g) of $X \in \mathcal{C}$ is a morphism $g : X \to D$ with $D \in \mathcal{D}$ that satisfies that, for every $g' : X \to D'$ with $D' \in \mathcal{D}$, there exists $k : D \to D'$ such that $g' = kg$. That is, $\operatorname{Hom}_{\mathcal{C}}(D, D') \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(g,D')} \operatorname{Hom}_{\mathcal{C}}(X, D')$ is surjective. A left $\mathcal{D}$-approximation (D, g) is *minimal* if whenever $a \in \operatorname{End}_{\mathcal{C}}(D)$ satisfies that $ag = g$, then a is an isomorphism.

iii) D is said to be *contravariantly finite* if every $X \in \mathcal{C}$ admits a right $\mathcal{D}$-approximation. Dually, $\mathcal{D}$ is *covariantly finite* if every $X \in \mathcal{C}$ admits a left $\mathcal{D}$-approximation. We say that $\mathcal{D}$ is *functorially finite* if it is both covariantly and contravariantly finite.

Lemma 1.34 (Wakamatsu's Lemma)**.** *[LZ20, Lemma 3.1] [Jør09, Lemma 2.1] Let $\mathcal{X}$ be a full, extension-closed subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. If there exists a conflation $Y \rightarrowtail X \overset{f}{\twoheadrightarrow} H$ where $X \xrightarrow{f} H$ is a minimal right $\mathcal{X}$-approximation of H, then $Y \in \mathcal{X}^{\perp_1}$.*

Proof. This results follows from [Jør09, Lemma 2.1] since $\mathcal{X}$ can be thought as en extension closed subcategory of the triangulated category $\mathcal{K}^b(\operatorname{proj}\Lambda)$. For a general extriangulated category, this statement is dual to [LZ20, Lemma 3.1]. Let $Y \rightarrowtail X \overset{f}{\twoheadrightarrow} H \dashrightarrow$ be a conflation where $X \xrightarrow{f} H$ is a minimal right $\mathcal{X}$-approximation of H. Let $X' \in \mathcal{X}$ consider the exact sequence induced by the functor $\operatorname{Hom}(X', -)$

$$\operatorname{Hom}(X', X) \xrightarrow{\operatorname{Hom}(X',f)} \operatorname{Hom}(X', H) \xrightarrow{\alpha} \mathbb{E}(X', Y) \xrightarrow{\beta} \mathbb{E}(X', X) \xrightarrow{\mathbb{E}(X',f)} \mathbb{E}(X', H).$$

Since f is a right $\mathcal{X}$-approximation, we know that $\operatorname{Hom}(X', f)$ is surjective and that $\alpha = 0$. We will show that $\mathbb{E}(X', f)$ is injective. Let $\delta \in \mathbb{E}(X', X)$ be realized by the conflation

$$X \overset{u}{\rightarrowtail} Z \overset{v}{\twoheadrightarrow} X' \overset{\delta}{\dashrightarrow}$$

such that $\mathbb{E}(X', f)(\delta) = 0$, that is, whenever we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Z \arrow[r, "v"] \arrow[d, "g"] & X' \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
H \arrow[r, "u'"] & L \arrow[r, "v'"] & X' \arrow[r, dashed, "\mathbb{E}(X',f)(\delta)"] & {}
\end{tikzcd}$$

where u' is a section and thus there exists w such that $wu' = 1$. Since $\mathcal{X}$ is extension closed, $Z \in \mathcal{X}$ and since f is a right $\mathcal{X}$-approximation, there exists $h : Z \to X$ such that $wg = fh$. We get that $f = wu'f = wgu = fhu$. Since f is right minimal, hu must be an isomorphism. In particular u is a section and $\delta = 0$. This implies that $\mathbb{E}(X', f)$ is injective and thus $\mathbb{E}(X', Y) = 0$ as wished. ☐

Remark 1.35. Suppose that $\mathcal{X}$ is additive as well. Given that $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is Hom-finite and satisfies WIC, if we have a conflation $Y \rightarrowtail X \overset{f}{\twoheadrightarrow} H$ where $X \xrightarrow{f} H$ is a right $\mathcal{X}$-approximation of H, we can find a conflation $Y' \rightarrowtail X' \overset{f'}{\twoheadrightarrow} H$ where f' is a right $\mathcal{X}$-approximation of H that is minimal. Indeed, by [Jør09, Lemma 4.1], there exists a direct summand X' of X and a minimal right $\mathcal{X}$-approximation $X' \xrightarrow{f'} H$. In particular, there exist $s : X \to X'$ such that $f's = f$. Since $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfies WIC and f is a deflation, then f' is one as well.

Definition 1.36. [INP24, Definition 2.1] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. An *almost split sequence* is a conflation $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$ such that:

i) δ is not split.

ii) For any morphism $h : X \to X'$ that is not a section, there exists a factorization

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[dl, "k"] \\
X' &
\end{tikzcd}.$$

iii) For any morphism $j : Z' \to Z$ that is not a retraction, there exists a factorization

$$\begin{tikzcd}
& Z' \arrow[dl, "l"'] \arrow[d, "j"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}.$$

We recall that $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ has Auslander-Reiten triangles. Consider the assignment $X \to \tau X$ given in the following way. Let $\mathcal{K}^{[-1,0]}(\operatorname{inj}\Lambda)$ be the homotopy category of 2-term complexes of injective modules. Recall that the Nakayama functor $\nu = D\operatorname{Hom}_\Lambda(-,\Lambda)$ induces an equivalence $\operatorname{proj}\Lambda \xrightarrow{\nu} \operatorname{inj}\Lambda$ and thus an equivalence

$$\nu : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \longrightarrow \mathcal{K}^{[-1,0]}(\operatorname{inj}\Lambda)$$

$$\begin{tikzcd} X^{-1} \arrow[d, "x"] & & \nu X^{-1} \arrow[d, "\nu x"] \\ X^0 \arrow[r, phantom, "\longmapsto"] & & \nu X^0 \end{tikzcd}$$

We let $\tau X = X_{H^{-1}(\nu X)}$ if $H^{-1}(\nu X) \neq 0$ and 0 otherwise. In particular, $H^0(\tau X) \simeq \tau H^0(X)$ for all $X \neq \begin{tikzcd}[row sep=small] P \arrow[d] \\ 0 \end{tikzcd}$ with $P \in \operatorname{proj}\Lambda$, where $\tau H^0(X)$ is the Auslander-Reiten translation of $H^0(X)$ in $\operatorname{mod}\Lambda$

Proposition 1.37. *The extriangulated category* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *has Auslander-Reiten conflations. That is, for any indecomposable non-projective* $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$*, there exists an almost split sequence*

$$\tau X \rightarrowtail X' \twoheadrightarrow X \dashrightarrow .$$

Furthermore, for all $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \setminus \operatorname{add}(\Lambda)[1]$, $H^0(\tau X) \simeq \tau H^0(X)$*, and for* $P \in \operatorname{add}(\Lambda)$, $H^0(\tau P[1]) \simeq \nu P$.

Proof. The proposition is a consequence of [INP18, Propositions 5.17 and 5.18]. Indeed, let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Given that ν gives rise to a relative Auslander-Reiten-Serre duality, there exists an almost split extension in $\mathbb{E}_{\mathcal{D}^b(\operatorname{mod}\Lambda)}(X, \nu X[-1])$. By [INP18, Proposition 5.17] and given that $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is functorially finite in $\mathcal{D}^b(\operatorname{mod}\Lambda)$, there exists an almost split extension $\delta \in \mathbb{E}_{[-1,0]}(X, Y)$, where Y is a direct summand of Y' and $Y' \to \nu X[-1]$ is a minimal right $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$-approximation of $\nu X[-1]$. Let

$$M = H^0(\nu X[-1]) = H^{-1}(\nu X)$$

and consider X_M its minimal projective presentation. Then we have a morphism in $D^b(\operatorname{mod}\Lambda)$ (actually, in $\mathcal{K}^b(\operatorname{mod}\Lambda)$ since $X_M \in \mathcal{K}^b(\operatorname{proj}\Lambda)$)

$$h : X_M \xrightarrow{\pi_M} M \xrightarrow{\iota_M} \nu X[-1]$$

where M is seen as a stalk complex in degree 0 and ι_M is given by the commutative diagram $\begin{tikzcd}[row sep=small] M \arrow[r, hook] \arrow[d] & \nu X^{-1} \arrow[d, "\nu x"] \\ 0 \arrow[r] & \nu X^0 \end{tikzcd}$. Consider any morphism $Z \xrightarrow{f} \nu X[-1]$ with $Z \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ represented by the following commutative diagram

$$\begin{tikzcd} Z^{-1} \arrow[r, "0"] \arrow[d, "z"] & 0 \arrow[d] \\ Z^0 \arrow[r, "f"] \arrow[d] & \nu X-1 \arrow[d, "\nu x"] \\ 0 \arrow[r] & \nu X^0. \end{tikzcd}$$

Since $\nu x\, f = 0$, then f factors as $Z^0 \xrightarrow{f} M \hookrightarrow \nu X^{-1}$. But Z^0 is projective and π_M is surjective, hence there is $g^0 : Z^0 \to X_M^0$ such that $\pi_M g^0 = f$. Given that $0 = f\, z = \pi_M(g^0 z)$, then $g^0 z$ restricts to a map $Z^{-1} \xrightarrow{g^0 z} \operatorname{Ker} \pi_M$. Using that Z^{-1} is projective and that $x_M : X_M^{-1} \to \operatorname{Ker} \pi_M$ is projective, we obtain $g^{-1} : Z^{-1} \to X_M^{-1}$ such that $g^0 z = x_M g^{-1}$. We have constructed a morphism $g : Z \to X_M$ such that $hg = f$ and thus $h : X_M \xrightarrow{\pi_M} M \xrightarrow{\iota_M} \nu X[-1]$ is a right $\mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$-approximation of $\nu X[-1]$. Since M is indecomposable, then so is $X_M = \tau X$ and hence h is minimal. Since Y' is a direct summand of X_M, then $Y' \simeq \tau X$. We conclude that there exists an almost split conflation

$$\tau X \rightarrowtail X' \twoheadrightarrow X \dashrightarrow . \qquad \square$$

Chapter 2

Semistability for finite-dimensional algebras

Throughout this chapter we suppose that $\Bbbk$ is algebraically closed field of characteristic 0, usually $\Bbbk = \mathbb{C}$. The goal of this chapter is to introduce and compare three notions of semistability in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$:

i) M-semistability, defined by the non-vanishing of the *determinantal semi-invariant* associated with $M \in \operatorname{mod}\Lambda$ (Definition 3.66, Definition 2.15);

ii) virtual semistability, characterized by the non-vanishing of a *virtual semi-invariant* of weight $\delta \in K_0(\operatorname{mod}\Lambda) \simeq \mathbb{Z}^n$ (Definition 2.2, Remark 2.19);

iii) numerical semistability, defined as a numerical condition on the pairing between the g-vector of a projective presentation and a weight $\delta \in K_0(\operatorname{mod}\Lambda)$. (Definition 2.21).

The main result of this chapter establishes the implication i) $\implies$ ii) $\implies$ iii). We begin in Section 2.1.1 by reviewing the framework provided by Geometric Invariant Theory (GIT) for the study of representation varieties as introduced by A. King in [Kin94]. In Section 2.1.3, we apply this framework to the varieties $\operatorname{Hom}_\Lambda(X^{-1}, X^0)$ of morphism between projective modules $X^{-1}, X^0 \in \operatorname{proj}\Lambda$ under the action of the group $\operatorname{Aut}_\Lambda(X^{-1}) \times \operatorname{Aut}_\Lambda(X^0)$, where we provide the tools needed to establish the implication i) $\implies$ iii) later in the chapter. In Section 2.2.2, we revisit the contributions of K. Igusa, K. Orr, G. Todorov, and J. Weyman in [IOTW09], where they introduced the concept of a *virtual space of presentations* along with the notion of a *virtual semi-invariant*. Additionally, we revisit the concept of a *determinantal semi-invariant* in this context.

Inspired by their work, we introduce in Section 2.2.2 the notion of *virtual semistability*, *M-semistability* and *numerical semistability*, and note that M-semistability implies virtual semistability. We end this chapter by showing that both virtual and M-semistability imply numerical semistability, and provide examples where the converse does not hold (Example 2.24). Additionally, we present an instance of a weight δ for which the associated subcategory of numerically δ-semistable projective presentations fails to be closed under extensions in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ (Example 2.25).

The findings presented in this chapter originate from [Gar23, Section 4]. Our results should be interpreted within an approach aimed at finding a notion of semistability that captures both geometric and homological properties akin to King's notion of semistability for $\operatorname{mod}\Lambda$. As shown by Example 2.25, numerical stability does not satisfactorily mirror the concept of θ-semistability, as the subcategory of θ-semistable modules in $\operatorname{mod}\Lambda$ is *wide* and thus closed under extensions. In Chapters 3 and 4, we will demonstrate that M-stability presents a more promising alternative, given its connection to τ-tilting theory.

2.1 Geometric Invariant Theory in representation theory

Geometric Invariant Theory (GIT) was developed by D. Mumford as a method for constructing quotients of an action of a reductive group G over a quasi-projective scheme R. He showed that one can find an open subvariety U of R such that the set of orbits of points in U (modulo an equivalence relation) defines a quotient in the category of schemes, which itself a projective scheme. The points of said open subvariety are known as *semistable points* and are identified via the use of *semi-invariants.* In this section, we review the fundamental concepts of GIT essential for understanding A. King's findings on GIT for varieties of modules over a finite-dimensional algebra Λ, which we revisit in Section 2.1.1. We then apply them to study vector spaces of morphism between projective Λ-modules. For a more comprehensive overview of GIT, see [MFK94, Hos16].

2.1.1 Semistability in $\operatorname{mod}\Lambda$

Let R be a quasi-projective $\mathbb{C}$-scheme equipped with the action of a an algebraic group G. A *character* of G is a morphism of algebraic groups $\chi : G \to \mathbb{C}^*$. An algebraic group morphism $\lambda : \mathbb{C}^* \to G$ is called a *one-parameter subgroup* or a *co-character* of G. For any character χ and one-parameter subgroup λ, the composition $\chi \circ \lambda : \mathbb{C}^* \to \mathbb{C}^*$ induces an integer paring $\langle -, - \rangle$ between the set of one-parameter subgroups and the character group. Indeed, since every algebraic group automorphism of $\mathbb{C}^*$ is of the form $t \mapsto t^m$ for some $m \in \mathbb{Z}$, we define $\langle \lambda, \chi \rangle$ to be the integer m such that $\chi \circ \lambda(t) = t^m$.

Definition 2.1. Let $\mathbb{C}[R]$ denote the ring of regular functions over R and let χ be a character of the group G. A *semi-invariant* $f \in \mathbb{C}[R]$ of weight χ, or χ-semi-invariant, is a regular function such that

$$f(g \cdot x) = \chi(g) f(x)$$

for all $g \in G$ and all $x \in R$.

For a non-trivial character χ, define the graded ring

$$SI(R)^{G,\chi} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}[R]^{G,\chi^m},$$

where the $\mathbb{C}[R]^{G,\chi^n}$ denote the set of semi-invariant functions over R of weight χ^n.

Definition 2.2. Let $x \in R$ and χ a character of G. We say that x is *χ-semistable* if there exist $m \in \mathbb{Z}_{>0}$ and $f \in \mathbb{C}[R]^{G,\chi^m}$ such that $f(x) \neq 0$. We denote by $R^{\chi,ss}$ the open subset of semistable points.

When G is a reductive group, Nagata's theorem implies that for every choice of character χ, the ring $SI(R)^{G,\chi}$ is finitely generated. Consequently, $\mathrm{Proj}(SI(R)^{G,\chi})$ is a projective variety, known as the projective GIT quotient of R by G, and denoted as $R//_{\chi}G$. The points in $R//_{\chi}G$ parametrize orbits of χ-semistable points up to S-equivalence[1].

Let Λ be a finite-dimensional basic $\mathbb{C}$-algebra and let $\mathrm{mod}\,\Lambda$ bet the category of right finite-dimensional Λ-modules. Since $\mathbb{C}$ is algebraically closed and of characteristic 0, there exists a finite quiver $Q = (Q_0, Q_1)$ such that $\Lambda \simeq \mathbb{C}Q/I$, where $\mathbb{C}Q$ is the path algebra associated to Q and I is an admissible ideal of relations of $\mathbb{C}Q$. The multiplication in $\mathbb{C}Q$ is induced by the composition of arrows in Q: for every diagram $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$, we denote $\beta\alpha$ the path from i to k following α and then β. A representation of a quiver with relations (Q, I) is given by a choice of a finite-dimensional $\mathbb{C}$-vector space V_i for each vertex $i \in Q_0$ and a linear map M_α for every arrow $\alpha \in Q_1$, such that the M_α satisfy the relation in I. We denote by $\mathrm{rep}(Q, I)$ the abelian category of representations of (Q, I), and recall that $\mathrm{mod}\,\mathbb{C}Q/I \simeq \mathrm{rep}(Q^{op}, I^{op})$, where (Q^{op}, I^{op}) is the quiver with relations whose underlying set of vertices is the same of that of (Q, I), but with all arrows pointing in the opposite direction. In this section, we adopt the language of representation theory of quivers for the sake of simplicity.

Let (Q, I) be a quiver with relations and let $n = |Q_0|$ be the number of vertices of Q. To any representation $M = (V_i, M_\alpha)_{i\in Q_0}^{\alpha\in Q_1}$, we associate the integer vector $\underline{\dim}(M) = (\dim_{\mathbb{C}}(V_i))_{i\in Q_0} \in \mathbb{Z}^n_{\geq 0}$. Fix $\delta = (\delta_i)_{i\in Q_0} \in \mathbb{Z}^n_{\geq 0}$, we denote by $R_\delta(Q, I)$ the subset of

$$R_\delta(Q) = \prod_{i \xrightarrow{\alpha} j} Mat_{\delta_j \times \delta_i}(\mathbb{C})$$

of $(M_\alpha)_{\alpha\in Q_1}$ that satisfy the relations given by I. Then $R_\delta(Q, I)$ is quasi-projective since is a closed subvariety of the affine variety $R_\delta(Q)$. Moreover, $R_\delta(Q, I)$ is equipped with a group action of

$$GL_\delta(\mathbb{C}) = \prod_{i\in Q_0} GL_{\delta_i}(\mathbb{C}),$$

given by $g \cdot (M_\alpha)_{\alpha\in Q_1} = (g_j M_\alpha g_i^{-1})_{i \xrightarrow{\alpha} j \in Q_1}$, for all $g = (g_i)_{i\in Q_0}$ and $(M_\alpha)_{\alpha\in Q_1} \in R_\delta(Q, I)$. Since $GL_\delta(\mathbb{C})$ is the product of finitely many reductive groups, it is itself reductive. Let χ be a character of $GL_\delta(\mathbb{C})$, then there exists $\theta \in \mathbb{Z}^n$ such that

$$\chi(g) = \prod_{i\in Q_0} det(g_i)^{\theta_i}$$

for all $g \in GL_\delta(\mathbb{C})$. For any $\theta \in \mathbb{Z}^n$, we denote by χ_θ the character described by the previous formula. We now present a Hilbert-Mumford type criterion for semistability introduced by A. King in [Kin94].

Proposition 2.3. *[Kin94, Proposition 2.5] Let R be a quasi-projective variety equipped with the action of a reductive group G. Let Δ be kernel of the G-action over R and le χ be a character of G. A point $x \in R$ is χ-semistable if and only if*

i) $\chi(\Delta) = \{1\}$

---

[1] Two points $x, y \in R$ are S-equivalent if and only if $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$.

ii) Every one-parameter subgroup λ *of* G*, for which* $\lim_{t\to 0}(\lambda(t)\cdot x)$ *exists, satisfies* $\langle \chi, \lambda\rangle \leq 0$.

One of King's most influential results is showing that when $R = R_\delta(Q, I)$ and $G = GL_\delta(\mathbb{C})$, Proposition 2.3 can be interpreted in the language of representation theory.

Proposition 2.4. *[Kin94, Propostion 3.1] Let* $\theta \in \mathbb{Z}^n$*,and let* χ *be the associated* $GL_\delta(\mathbb{C})$*-character. Let* $x_M \in R_\delta(Q, I)$ *be a point corresponding to a representation* $M \in \operatorname{rep}(Q, I)$*. Then* x_M *is* χ_θ*-semistable if and only if* M *satisfies the following conditions*

i) $\langle \theta, \underline{\dim}(M)\rangle = 0$*;*

ii) For every subrepresentation $N \subset M$*,* $\langle \theta, \underline{\dim}(N)\rangle \leq 0$*.*

If M *is a representation satisfying conditions i) and ii), we say that it is* θ*-semistable.*

Note that for any representation $M \in \operatorname{rep}(Q, I)$, we have that $\underline{\dim}(M) = [M] \in K_0(\operatorname{rep}(Q, I))$, thus the previous definition translates straightforwardly to the language of modules over a finite-dimensional algebra Λ. We denote by $\mathscr{W}_\theta \subset \operatorname{mod}\Lambda$ the full subcategory whose objects are those who are θ-semistable. The following result was observed by King in [Kin94] as well.

Proposition 2.5. *[Kin94] Let* $\theta \in \mathbb{Z}^n$*, then* $\mathscr{W}_\theta$ *is closed under direct summands, extensions, kernels and cokernels. In other words, it is a wide subcategory (see Definition 3.12).*

Proof. Let $\theta \in \mathbb{Z}^n$. We are going to prove that $\mathscr{W}_\theta$ is closed under extensions. Consider the short exact sequence $0 \to M \to N \to L \to 0$, where M and L are θ-semistable. Since $[N] = [M] + [L]$ and $\langle \theta, [M]\rangle = \langle \theta, [L]\rangle = 0$, we get that $\langle \theta, [N]\rangle = 0$. Let $0 \neq N' \subset N$, we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f"] & N' \arrow[r] \arrow[d, hook] & L' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r, "\iota"] & N \arrow[r, "\pi"] & L \arrow[r] & 0
\end{tikzcd}.$$

where $M' \simeq \iota^{-1}(N')$ and $L' \simeq \pi(N')$ and g and f are monomorphisms. Since M, L are θ-semistable $\langle \theta, [N']\rangle = \langle \theta, [M']\rangle + \langle \theta, [L']\rangle \leq 0$. We have showed that N is θ-semistable, and hence $\mathscr{W}_\theta$ is closed under extensions.

Let us know prove that $\mathscr{W}_\theta$ is close under kernels. Let $f : M \to N$ be a morphism between θ-semistable modules. Then for any $0 \neq M' \subset \operatorname{Ker} f \subset M$ we have that $\langle \theta, [M']\rangle \leq 0$. Given that we have a short exact sequence $0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0$, we have that $0 = \langle \theta, [M]\rangle = \langle \theta, [\operatorname{Ker} f]\rangle + \langle \theta, [\operatorname{Im} f]\rangle$. But since $\operatorname{Ker} f \subset M$ and $\operatorname{Im} f \subset N$, both $\langle \theta, [\operatorname{Ker} f]\rangle$ and $\langle \theta, [\operatorname{Im} f]\rangle$ are non-positive, thus $\langle \theta, [\operatorname{Im} f]\rangle = \langle \theta, [\operatorname{Ker} f]\rangle = 0$. In particular, $\operatorname{Ker} f$ is θ-semistable. To see that $\operatorname{Coker} f$ is θ-semistable, note that $\langle \theta, [\operatorname{Coker} f]\rangle = \langle \theta, [N]\rangle - \langle \theta, [\operatorname{Im} f]\rangle = 0$. Moreover, for any $L \subset \operatorname{Coker} f$, there exists $\operatorname{Im} f \subset N' \subset N$ such that $L \simeq N'/\operatorname{Im} f$ and hence

$$\langle \theta, [L]\rangle = \langle \theta, [N']\rangle - \langle \theta, [\operatorname{Im} f]\rangle = \langle \theta, [N']\rangle \leq 0.$$

□

2.1.2 Determinantal semi-invariants I

King's groundbreaking results were developed concurrently to the effort of classifying all semi-invariants of the action of $GL_\delta(\mathbb{C})$ over $R_\delta(Q,I)$. These efforts culminated in the discovery of generators for the semi-invariant ring $SI(R_\delta(Q,I))^{GL_\delta(\mathbb{C}),\chi_\theta}$ for a given $\theta \in \mathbb{Z}^n$. Let Λ be the finite-dimensional algebra such that $\operatorname{rep}(Q,I) \simeq \operatorname{mod}\Lambda$ and recall that $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) = \oplus_{i=1}^n \mathbb{Z}[P_i] \simeq \mathbb{Z}^n$.

Definition 2.6. [Sch91] Let $X^{-1}, X^0 \in \operatorname{proj}\Lambda$, $x \in \operatorname{Hom}_\Lambda(X^{-1}, X^0)$ and $M \in \operatorname{mod}\Lambda$ such that $\langle [X^0] - [X^{-1}], [M]\rangle = 0$, that is, such that

$$\dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^{-1}, M)) = \dim_{\Bbbk}(\operatorname{Hom}_\Lambda(X^0, M)).$$

We define the *determinantal semi-invariant* associated with x as the regular function, which we denote by $s(x,-)$, such that for any $m \in R_{[M]}(Q,I)$, we have

$$s(x,m) = s(x,M) = \det\left(\operatorname{Hom}_\Lambda(X^0, M) \xrightarrow{\operatorname{Hom}_\Lambda(x,M)} \operatorname{Hom}_\Lambda(X^{-1}, M)\right).$$

Remark 2.7. Let $M \in \operatorname{mod}\Lambda$ and let $[M]$ be its class in $K_0(\operatorname{mod}\Lambda)$. Note that the value of the map $s(-,M)$ depends on the choice of basis for the $\operatorname{Hom}_\Lambda(X^0, M)$ and $\operatorname{Hom}_\Lambda(X^{-1}, M)$ spaces. However, we mostly care about the non-annihilation of these functions, and since base change doesn't affect this property, the choice of basis is mostly omitted.

The following theorem holds for all finite-dimensional algebras over a algebraically closed field $\Bbbk$ of characteristic 0, but we state it here for the case when $\Bbbk = \mathbb{C}$.

Theorem 2.8. *[DW00, SVdB01, Dom02] Let $\delta \in \mathbb{Z}^n_{\geq 0}$ and $\theta \in \mathbb{Z}^n$ be integer vectors such that $\langle \theta, \delta\rangle = 0$. Then, the ring $SI(R_\delta(Q,I))^{GL_\delta(\mathbb{C}),\chi_\theta}$ is generated by all determinantal semi-invariants $s(x,-)$ where $x \in \operatorname{Hom}_\Lambda(X^{-1}, X^0)$ for some $X^{-1}, X^0 \in \operatorname{proj}\Lambda$ such that $[X^0] - [X^{-1}] = l\theta$ with $l \in \mathbb{Z}_{\geq 0}$.*

Theorem 2.8 provides and alternative (but equivalent) criterion to that of King to determine whether a module Λ is θ-semistable. In Remark 2.14, we will see that for a given $X = \begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^0 \end{smallmatrix} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with $\langle [X], \delta\rangle = 0$, the non-vanishing of the semi-invariant $s(x,-)$ over a module M of dimension vector δ depends only in the isomorphism class of X in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. This implies that M is θ-semistable if and only if there exists $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with $[X] = l\theta$ for $l \in \mathbb{Z}_{\geq 0}$ such that $s(X,M) \neq 0$. We rephrase this statement in terms of the full subcategories $\mathscr{W}(X) = \{M \in \operatorname{mod}\Lambda \mid s(X,M) \neq 0\} \subset \operatorname{mod}\Lambda$ (see Section 3.2.3).

Proposition 2.9. *[Kin94, DW00, SVdB01, Dom02] Let $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$, then*

$$\mathscr{W}_\theta = \bigcup_{\substack{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \\ [X] \in \mathbb{Z}_{\geq 0}\theta}} \mathscr{W}(X),$$

where $\mathscr{W}(X) = \{M \in \operatorname{mod}\Lambda \mid s(X,M) \neq 0\}$.

Proof. Let M be θ-semistable module. By Proposition 2.4, M is (geometrically) χ_θ-semistable, in particular, there exists a χ_θ-semi-invariant f such that $f(M) \neq 0$. Since all χ_θ-semi-invariants are generated by determinantal semi-invariants by Theorem 2.8, there exists $X = \begin{array}{c} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^0 \end{array} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $[X] = l\theta$ for some $l \in \mathbb{Z}_{\geq 0}$ and $s(X, M) \neq 0$. We get that for every $M \in \mathscr{W}_\theta$, there is X such that $M \in \mathscr{W}(X)$, and so $\mathscr{W}_\theta \subset \bigcup_{\substack{X\in\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \\ [X]\in\mathbb{Z}_{\geq 0}\theta}} \mathscr{W}(X)$. The other inclusion follows from the fact that $s(X, -)$ is a semi-invariant of weight $[X] = l\theta$ for some $l \in \mathbb{Z}_{\geq 0}$. □

The following result shows the relation between semistability in $\operatorname{mod}\Lambda$ and τ-tilting theory. It has been proved in full generality in [Yur18, BST19]. We include here a proof when Λ is an algebra over a algebraically closed field, to showcase the relevance of the $s(X, -)$ invariants, and why one could be brought to define semistability in their terms. A representation-theoretical proof, based on that in [Yur18], is presented in Section 3.1.2.

Theorem 2.10. *[Yur18, BST19] Let $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a presilting complex (Definition 3.27). Then*

$$\mathscr{W}_{[U]} = {}^{\perp}H^{-1}(\nu U) \cap H^0(U)^{\perp} = \mathscr{W}(U).$$

Proof. Let $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a presilting complex. By Proposition 2.9, we have that $\mathscr{W}(U) \subset \mathscr{W}_{[U]}$. To prove that $\mathscr{W}(U)$ contains $\mathscr{W}_{[U]} = \bigcup_{\substack{X\in\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \\ [X]\in\mathbb{Z}_{\geq 0}[U]}} \mathscr{W}(X)$, consider $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $[X] = l[U]$ for some $l \in \mathbb{Z}_{\geq 0}$ and let $M \in \mathscr{W}(X)$. Then, there exist $x \in \operatorname{Hom}_\Lambda(X^{-1}, X^0) = R$ such that $X \simeq \begin{array}{c} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^0 \end{array}$ and $s(x, M) \neq 0$. Since this condition is open, the generic point $\eta \in R$ must satisfy that $s(\eta, M) \neq 0$. Because U is presilting, we can choose a representative of its isomorphism class $\begin{array}{c} U^{-1} \\ \downarrow{\scriptstyle u} \\ U^0 \end{array}$, such that there are no non-zero common projective direct summands between U^{-1} and U^0. In particular, we can suppose that there is $P \in \operatorname{proj}\Lambda$ such that $X^i = (U^i)^{\oplus l} \oplus P$ for $i \in \{-1, 0\}$. Let $s'(-, M)$ be the determinantal semi-invariant defined by M on $R' = \operatorname{Hom}_\Lambda((U^{-1})^{\oplus l}, (U^0)^{\oplus l})$. Since $s'(\eta^{\oplus l}, M) \neq 0$, we get that $s'(\eta', M) \neq 0$, where η' is the generic point in R'. By a Dehy-Keller argument [DK08, Section 2.1], we know that since $U^{\oplus l}$ is a 2-term presilting complex, the orbit $\mathcal{O}_{u^{\oplus l}}$ inside R' must be open and dense. In particular, $\mathcal{X} \cap \mathcal{O}_{u^{\oplus l}} \neq \emptyset$, where $\mathcal{X} = \{y \in R' \mid s'(y, M) \neq 0\}$. Then, there must exist $u' \in \mathcal{O}_{u^{\oplus l}}$ such that $s(u', M) \neq 0$. Since U is a direct summand of $U^{\oplus l} \simeq \begin{array}{c} {U^{-1}}^{\oplus l} \\ \downarrow{\scriptstyle u'} \\ {U^0}^{\oplus l} \end{array}$, we must have that $s(u, M) \neq 0$. We get that $\mathscr{W}(X) \subset \mathscr{W}(U)$ for all X such that $[X] = l[U]$ for $l \in \mathbb{Z}_{\geq 0}$, which implies that

$$\mathscr{W}(U) = \mathscr{W}_{[U]}.$$

To show that $\mathscr{W}(U) = {}^{\perp}H^{-1}(\nu U) \cap H^0(U)^{\perp}$, remark that for any $U = \begin{array}{c} U^{-1} \\ \downarrow{\scriptstyle u} \\ U^0 \end{array} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $M \in \operatorname{mod}\Lambda$, the condition $s(u, M) \neq 0$ is equivalent to $\operatorname{Hom}_\Lambda(u, M)$

being an isomorphism. By Proposition 3.24, $\mathrm{Hom}_\Lambda(u, M)$ is an isomorphism if and only if $M \in {}^{\perp}H^{-1}(\nu U) \cap H^0(U)^{\perp}$, which gives the result. □

2.1.3 Semi-invariants of projective presentations

From now on we fix $\theta^{-1}, \theta^0 \in \mathbb{Z}^n_{\geq 0}$ as well as $X^{-1} = \bigoplus_{i=0}^n P_i^{\oplus \theta_i^{-1}}$ and $X^0 = \bigoplus_{i=0}^n P_i^{\oplus \theta_i^0}$ two projective modules in $\mathrm{mod}\,\Lambda$. Let

$$\begin{aligned} R(X^{-1}, X^0) &:= \mathrm{Hom}_\Lambda(X^{-1}, X^0) \\ A(X^{-1}, X^0) &:= \mathrm{End}_\Lambda(X^{-1})^{op} \times \mathrm{End}_\Lambda(X^0) \\ G(X^{-1}, X^0) &:= \mathrm{Aut}_\Lambda(X^{-1})^{op} \times \mathrm{Aut}_\Lambda(X^0) \subset A(X^{-1}, X^0). \end{aligned}$$

The group $G(X^{-1}, X^0)$ acts on the affine space $R(X^{-1}, X^0)$ via simultaneous multiplication: let $g = (g_{-1}, g_0) \in G(X^{-1}, X^0)$, where $g_{-1} \in \mathrm{Aut}(X^{-1})$, $g_0 \in \mathrm{Aut}(X^0)$ and $x : X^{-1} \to X^0$, then $g \cdot x = g_0 \cdot x \cdot g_{-1}$. Note that this is a well defined action since we chose to work in $\mathrm{Aut}(X^{-1})^{op}$. If we were to consider $\mathrm{Aut}(X^{-1})$ instead, some sign conventions would have to be adjusted. When the context allows it, we will write R instead of $R(X^{-1}, X^0)$ and do the same for A and G.

In this section, we study the action of $G(X^{-1}, X^0)$ over the vector space $R(X^{-1}, X^0)$ withing the GIT framework for representations introduced by King, and establish a technical result that will be used in the rest of the chapter. It's worth noting that classical GIT tools cannot be directly applied here since the algebraic group $G(X^{-1}, X^0)$ is not reductive in general. However, recent developments in non-reductive GIT and moduli spaces, which are beyond the scope of this text, provide promising new directions in the aim of defining stability for projective presentations [DK07, BDF$^+$22, AHLH23].

As in Section 2.1.1, we wish to study the characters, one-parameter subgroups and semi-invariants associated to the action of $G(X^{-1}, X^0)$ over $R(X^{-1}, X^0)$. Recall that any algebraic group G over $\mathbb{C}$ satisfies that $G = U \rtimes G_{red}$ where U is its unipotent radical and G_{red} is reductive [Hoc12, Theorem 4.3]. We wish to describe U and G_{red} when $G = G(X^{-1}, X^0)$. Note that $A = A(X^{-1}, X^0)$ is always a finite-dimensional $\mathbb{C}$-algebra, and thus its radical N coincides with its nil-radical. By the Wedderburn-Artin theorem, we have that

$$A/N \cong \bigoplus_{i=1}^n \left(M_{\theta_i^{-1}}(D_i) \times M_{\theta_i^0}(D_i)^{op} \right)$$

where $N = \mathrm{rad}(\mathrm{End}(X^{-1})) \times \mathrm{rad}(\mathrm{End}(X^0))$ and $D_i = \mathrm{End}(P_i)/\,\mathrm{rad}(\mathrm{End}(P_i)) \simeq \mathbb{C}$. Using the fact that $f \in A$ is invertible if and only if its image in A/N is invertible, we get that

$$G = G(X^{-1}, X^0) = (1_A + N) \rtimes \left(\prod_{i=1}^n \left(GL_{\theta_i^{-1}}(D_i) \times GL_{\theta_i^0}(D_i) \right) \right).$$

Let $U = 1_A + N$, then U is a normal subgroup of G and, by definition, all its elements are unipotent. Since Λ is finite-dimensional, there exists $m \in \mathbb{Z}_{>0}$ and a series of subgroups

$$1_A \trianglelefteq 1_A + N^m \trianglelefteq \cdots \trianglelefteq 1_A + N^2 \trianglelefteq 1_A + N = U$$

where $1+N^i$ is a normal subgroup of $1+N^{i-1}$, $N^{m+1}=0$, and such that every quotient is abelian. We conclude that U is solvable and hence, $U=1_A+N$ is the unipotent radical of G, since it is closed and connected.

We conclude that

$$G(X^{-1},X^0)_{red}\cong\prod_{i=1}^{n}\Big(GL_{\theta_i^{-1}}(D_i)\times GL_{\theta_i^0}(D_i)\Big),$$

where $D_i\cong\mathbb{C}$.

Lets now compute all $G(X^{-1},X^0)$-characters. Observe that if G is unipotent, then every character is trivial. Indeed, if $\chi:G\to\mathbb{C}^*$ is a algebraic group morphism from an unipotent group G, then $\chi(G)$ is unipotent. But any unipotent subgroup of $\mathbb{C}^*$ is trivial, so $\chi\equiv 1$. In particular, when $G=U\rtimes G_{red}$, the set of characters of G can be identified with that of G_{red}. In our case, every character

$$\chi:\prod_{i=1}^{n}\Big(GL_{\theta_i^{-1}}(\mathbb{C})\times GL_{\theta_i^0}(\mathbb{C})\Big)\to\mathbb{C}^*$$

is determined by integer vectors d^{-1} and $d^0\in\mathbb{Z}^n$ such that

$$\chi\left((g_{-1}^i,g_0^i)_{1\le i\le n}\right)=\prod_{i=1}^{n}\det(g_{-1}^i)^{d_i^{-1}}\cdot\det(g_0^i)^{d_i^0}.$$

Hence, the group of characters of G is isomorphic to $\mathbb{Z}^{2n}$. For $\bar{d}=(d^{-1},d^0)\in\mathbb{Z}^{2n}$ we will denote by $\chi_{\bar{d}}$ the character given by the previous formula.

All one-parameter subgroups of $G(X^{-1},X^0)$ can also be described in terms of U and G_{red}: they are all of the form $u\hat{\lambda}u^{-1}$ for $u\in U$ and $\hat{\lambda}:\mathbb{C}^*\to G_{red}$. Indeed, let $H=ker(\lambda)=\lambda^{-1}(1_G)$, given that H is a closed subgroup of $\mathbb{C}^*$ and that $\mathbb{C}^*$ is reductive, we have that $\lambda(\mathbb{C}^*)\cong\mathbb{C}^*/H$ is a reductive subgroup of $G=U\rtimes G_{red}$. As we are working in characteristic 0, by [Hoc12, Proposition 4.2], there exists $u\in U$ such that $u^{-1}\lambda(\mathbb{C}^*)u\le G_{red}$. Then $\hat{\lambda}=u^{-1}\lambda u:\mathbb{C}^*\to G_{red}$ satisfies the property. Moreover, since G_{red} is a product of $GL_k(\mathbb{C})$'s, all one-parameter subgroups are of the form $\lambda=u\tilde{\lambda}u^{-1}$, where $\tilde{\lambda}$ is a one-parameter subgroup with image in a maximal torus of G_{red}.

Let $x\in R=R(X^{-1},X^0)$ and consider its orbit $G\cdot x\subset R$. Note that $G\cdot x$ can be identified with the isomorphism class of x as an object in $C^{[-1,0]}(\operatorname{proj}\Lambda)$. The following proposition will give us a link between inflations in $C^{[-1,0]}(\operatorname{proj}\Lambda)$ and semistability. It is a partial analog to King's numerical condition for module semistability (Proposition 2.4) and it will be the key to show that both M-stability and virtual stability imply numerical stability for projective presentations (see Proposition 2.22).

Proposition 2.11. *Let* $x\in R$ *and let* $X=\begin{matrix}X^{-1}\\ \downarrow x\\ X^0\end{matrix}\in C^{[-1,0]}(\operatorname{proj}\Lambda)$. *If* x *if* χ-*semistable, then*

i) $\langle(-[X^{-1}],[X^0]),\chi\rangle=0$;

ii) For any inflation $Y\rightarrowtail X$ *in* $C^{[-1,0]}(\operatorname{proj}\Lambda)$, *we must have that*

$$\langle(-[Y^{-1}],[Y^0]),\chi\rangle\ge 0.$$

Proof. Suppose $x \in R$ is $\chi_{\bar{d}}$-semistable for some $\bar{d} \in \mathbb{Z}^{2n}$. Let $G_0 = \{g \,|\, g \cdot x = x \quad \forall x \in R\}$. Let $f \in \mathbb{C}[R]^{G,\chi^m_{\bar{d}}}$ such that $f(x) \neq 0$ with $m \geq 1$, then $f(x) = f(g \cdot x) = \chi(g)^m_{\bar{d}} f(x)$ for any $g \in G_0$. That is, $\chi^m_{\bar{d}}(G_0) \equiv 1$. In particular, for $\Delta = \left\{ (t^{-1} \cdot 1_{\theta_i^{-1}}, t \cdot 1_{\theta_i^0}) \mid t \in \mathbb{C}^* \right\}_{1 \leq i \leq n} \cong \mathbb{C}^* \subseteq G_0$, we must have that

$$\chi^m(\Delta) = \left(\prod_{i=1}^{n} \det(t^{-1} \cdot 1_{\theta_i^{-1}})^{d_i^{-1}} \det(t \cdot 1_{\theta_i^0})^{d_i^0} \right)^m = \\ t^{m\left(-\sum_{i=1}^n \theta_i^{-1} d_i^{-1} + \sum_{i=1}^n \theta_i^0 d_i^0\right)} = 1$$

which in turn implies that

$$\langle (-[X^{-1}], [X^0]), (d^{-1}, d^0) \rangle = \langle (-[X^{-1}], [X^0]), \chi \rangle = -\sum_{i=1}^{n} \theta_i^{-1} d_i^{-1} + \sum_{i=1}^{n} \theta_i^0 d_i^0 = 0.$$

We prove now that for any inflation $Y \rightarrowtail X$ in $C^{[-1,0]}(\operatorname{proj}\Lambda)$, $\langle (-[Y^{-1}], [Y^0]), \chi \rangle \geq 0$. Let f be a χ^m-semi-invariant for an $m \leq 1$. If λ is a one-parameter subgroup of G, we must have that for every $t \in \mathbb{C}^*$,

$$f(\lambda(t) \cdot x) = \chi^m(\lambda(t)) f(x) = t^{m\langle \lambda, \chi \rangle} f(x).$$

Suppose that $\lim_{t \to 0} \lambda(t) \cdot x$ exists and it is equal to $x' \in R$, then

$$f(x') = \lim_{t \to 0} t^{m\langle \chi, \lambda \rangle} f(x).$$

Since $f(x) \neq 0$, we must have that $\langle \lambda, \chi \rangle \geq 0$ for the above limit to exists. The statement in ii) will follow from proving that one-parameter subgroups such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists correspond to inflations of X in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$. As we have seen before, $\lambda(t) = u\tilde{\lambda}(t)u^{-1}$ where $\tilde{\lambda}$ is a one-parameter subgroup with image in a maximal torus of G and $u \in U$. Explicitly,

$$\lambda(t) = (\lambda_{-1}(t), \lambda_0(t)) = (g_{-1}\ \tilde{\lambda}_{-1}(t)\ (g_{-1})^{-1}, g_0\ \tilde{\lambda}_0(t)\ g_0^{-1}) =$$

$$= \left(g_{-1} \left(\begin{matrix} \begin{pmatrix} t^{\lambda^{-1}_{1,1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\lambda^{-1}_{\theta_1^{-1},1}} \end{pmatrix} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} t^{\lambda^{-1}_{1,i}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\lambda^{-1}_{\theta_i^{-1},i}} \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \begin{pmatrix} t^{\lambda^{-1}_{1,n}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\lambda^{-1}_{\theta_n^{-1},n}} \end{pmatrix} \end{matrix} \right. \right) (g_{-1})^{-1},$$

$$g_0 \begin{pmatrix} \begin{pmatrix} t^{\lambda^0_{1,1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\lambda^0_{\theta^0_1,1}} \end{pmatrix} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} t^{\lambda^0_{1,i}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\lambda^0_{\theta^0_i,i}} \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \begin{pmatrix} t^{\lambda^0_{1,n}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\lambda^0_{\theta^0_n,n}} \end{pmatrix} \end{pmatrix} g_0^{-1} \Bigg)$$

where $\lambda_{-1}(t),\ g_{-1} \in \mathrm{Aut}(X^{-1})$, and $\lambda_0(t),\ g_0 \in \mathrm{Aut}(X^0)$ for every $t \in \mathbb{C}^*$. Here, $\lambda^{\varepsilon}_{l,i}$ is the weight corresponding to the l-th copy of the indecomposable projective P_i inside of X^{ε} with $\varepsilon \in \{-1, 0\}$, $1 \leq i \leq n$ and $1 \leq l \leq \theta^{\varepsilon}_i$. We get that, for $\varepsilon \in \{-1, 0\}$, $X^{\varepsilon} = \bigoplus_{m\in\mathbb{Z}} X^i_m$ where each X^i_m is the direct sum of indecomposable projective summands Q of X^{ε} such that $\lambda_{\varepsilon}(t)(Q) = t^m Q$. So, for any $m, n \in \mathbb{Z}$, we have the following commutative diagram :

$$\begin{tikzcd} X^{-1}_n \arrow[r, hook] \arrow[d, "t^{n+m}\left(\pi^0_m \cdot x|_{X^{-1}_n}\right)"] & X^{-1} \arrow[d, "\lambda_0(t)\cdot x\cdot\lambda_{-1}(t)"] \\ X^0_m & X^0 \arrow[l, two heads, "\pi^0_m"] \end{tikzcd}$$

Since the limit when $t \to 0$ exists, then $\pi^0_m \cdot x|_{X^{-1}_n}$ must be zero when $n + m < 0$. Let $X^{-1}_{\leq n} = \bigoplus_{i\leq n} X^{-1}_{-i}$ and $X^0_{\leq n} = \bigoplus_{i\leq n} X^0_i$. Then, for every $n \in \mathbb{Z}$, x defines the object $X_{\leq n} := \begin{tikzcd} X^{-1}_{\leq n} \arrow[d, "x_{\leq n}"] \\ X^0_{\leq n} \end{tikzcd}$, which makes the following diagram commute

$$\begin{tikzcd} X^{-1}_{\leq n} \arrow[r, tail] \arrow[d, "x_{\leq n}"] & X^{-1} \arrow[d, "x"] \\ X^0_{\leq n} \arrow[r, tail] & X^0 \end{tikzcd}$$

Here $x_{\leq n} = \pi^0_m \cdot x|_{X^{-1}_n}$ when $m + n \geq 0$ and equals 0 otherwise. This gives us a sequence $0 \rightarrowtail \cdots \rightarrowtail X_{\leq n} \rightarrowtail X_{\leq n+1} \rightarrowtail \cdots \rightarrowtail X$ of inflations for X. Note that $[X^i] = [\bigoplus X^i_{\leq n+1}/X^i_{\leq n}] = \sum_{n\in\mathbb{Z}}[X^i_{\leq n+1}/X^i_{\leq n}]$ for $i \in \{-1, 0\}$. For a projective module Q and $1 \leq j \leq n$, denote by $[Q]_j$ the number of times the indecomposable projective P_j appears as a direct summand of Q. We can express the value of $\langle\lambda, \chi\rangle$ as

$$\langle\lambda, \chi\rangle =$$

$$\left(\sum_{j=1}^{n} \theta^{-1}_j \left(\sum_{m\in\mathbb{Z}} (-m)[X^{-1}_{\leq m}/X^{-1}_{\leq m-1}]_j\right)\right) + \left(\sum_{j=1}^{n} \theta^0_j \left(\sum_{m\in\mathbb{Z}} m[X^0_{\leq m}/X^0_{\leq m-1}]_j\right)\right) =$$

$$= \sum_{m\in\mathbb{Z}} m\langle \left(-[X^{-1}_{\le m}/X^{-1}_{\le m-1}], [X^{0}_{\le m}/X^{0}_{\le m-1}]\right), \chi\rangle =$$
$$= \sum_{m\in\mathbb{Z}} \langle \left(-[X^{-1}_{\le m}], [X^{0}_{\le m}]\right), \chi\rangle.$$

That is, the value of the paring between χ and λ is given by the inner product between the associated integer vector of χ and the classes in $\mathcal{K}_0(\operatorname{proj}\Lambda)$ of the projective modules $X^i_{\le m}$ for $i \in \{-1, 0\}$.

Given an object $Y = \begin{tikzcd} Y^{-1} \arrow[d, "y"] \\ Y^0 \end{tikzcd}$ that is the source of an inflation to X, we construct a one-parameter subgroup λ_Y such that it's associated filtration is $0 \hookrightarrow Y \hookrightarrow X$. Suppose $Y^{-1} = \bigoplus_{i=0}^n P_i^{\theta_i'^{-1}}$ and $Y^0 = \bigoplus_{i=0}^n P_i^{\theta_i'^{0}}$ with $\theta_i'^0 \le \theta_i^0$ and $\theta_i'^{-1} \le \theta_i^{-1}$ for all $1 \le i \le n$. Up to isomorphism, we can suppose $Y^i \subseteq X^i$. Let

$$\lambda_Y(t) = \left(\prod_{i=1}^n \operatorname{diag}_{\theta_i^{-1}}(1, \cdots, 1, t^{-1}, \cdots, t^{-1}), \prod_{i=1}^n \operatorname{diag}_{\theta_i^{0}}(1, \cdots, 1, t, \cdots, t)\right)$$

where each diagonal matrix has $\theta_i'^{-1}$ and $\theta_i'^0$ 1's respectively. Then $X^{-1}_{\le 0} = Y^{-1}$, $X^0_{\le 0} = Y^0$, $X^{-1}_{\le i} = X^0_{\le i} = 0$ for all $i < 0$ and $X^{-1}_{\le i} = X^{-1}$ and $X^0_{\le i} = X^0$ for all $i > 0$. Since x is semistable,

$$\langle \lambda_Y, \chi\rangle = \sum_{i<0} \langle \left(-[X^{-1}_{\le i}], [X^0_{\le i}]\right), \chi\rangle + \langle \left(-[Y^{-1}], [Y^0]\right), \chi\rangle +$$
$$\sum_{i>0} \langle \chi, \left(-[X^{-1}_{\le i}], [X^0_{\le i}]\right)\rangle = 0 + \langle(-[Y^{-1}], [Y^0]), \chi\rangle + 0 \ge 0.$$

□

Remark 2.12. Note that if every inflation satisfies ii) from the previous proposition, then $\langle \lambda, \chi\rangle \ge 0$ for every one-parameter subgroup λ such that the limit $\lim_{t\to 0} \lambda(t)\cdot x$ exists. If G were reductive, this would imply that x is χ-semistable as in [Kin94]. In Example 2.24 we present an instance where the converse does not hold.

2.2 Towards a notion of semistability in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$

As in [Kin94], one would like there to exist a *numerical* notion of semistability in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Before exploring this idea, we revisit determinantal semi-invariants and virtual presentation spaces following [IOTW09, IOTW15]. As in the previous section, $R = R(X^{-1}, X^0)$ and $G = (X^{-1}, X^0)$.

2.2.1 Determinantal semi-invariants II

To study whether Proposition 2.11 has a partial converse, one could try to explicitly describe its ring of semi-invariants $SI(R)^{G,\chi}$ for a given character χ.

Proposition 2.13. *[IOTW09, Proposition 5.1.3] Let $X^{-1}, X^0 \in \operatorname{proj}\Lambda$ and $M \in \operatorname{mod}\Lambda$. The regular function*

$$s(x, M) = \det\left(\operatorname{Hom}_\Lambda(X^0, M) \xrightarrow{\operatorname{Hom}_\Lambda(x,M)} \operatorname{Hom}_\Lambda(X^{-1}, M)\right)$$

is a $G(X^{-1}, X^0)$*-semi-invariant over* $R(X^{-1}, X^0)$ *with associated character* $\chi_{([M],[M])}$*, where* $([M],[M]) \in \mathbb{Z}^{2n}$*.*

Proof. By hypothesis $\dim_{\Bbbk} \operatorname{Hom}_\Lambda(X^0, M) = \dim_{\Bbbk} \operatorname{Hom}_\Lambda(X^{-1}, M)$, thus $s(x, M)$ is well defined for any $x \in R(X^{-1}, X^0)$. Let $g = (g_{-1}, g_0) \in G(X^{-1}, X^0) = \operatorname{Aut}(X^{-1}) \times \operatorname{Aut}(X^0)$, then

$$s(g \cdot x, M) = \det\left(\operatorname{Hom}_\Lambda(X^0, M) \xrightarrow{(g_0 \cdot x \cdot g_{-1})^*} \operatorname{Hom}_\Lambda(X^{-1}, M)\right) =$$
$$= \det(g_0^*) \cdot s(x, M) \cdot \det(g_{-1}^*).$$

The regular function $\chi(g) = \det(g_0^*) \cdot \det(g_{-1}^*)$ defines a character for the action of $G(X^{-1}, X^0)$, and as such, it factors through $G(X^{-1}, X^0)_{red}$, that is, $\chi(g) = \chi(g_{red}) = \det((g_0)^*_{red}) \cdot \det((g_{-1})^*_{red})$ since χ is trivial over the unipotent radical of $G(X^{-1}, X^0)$ as seen in Section 2.1.3. Recall $(g_0)_{red} = (g_0^i) \in \prod_{i=1}^n GL_{\theta_i^0}(D_i)$. Since $\operatorname{Hom}_\Lambda(P_i, M) \cong M_i \ \forall\ 1 \leq i \leq n$, where M_i is the vector space in the vertex i associated to M, we have that $(g_0)^*_{red}$ is a block-diagonal matrix in which the block corresponding to g_0^i appears $\dim M_i$ times. Thus, $\det((g_0)^*_{red}) = \prod_{i=1}^n \det(g_0^i)^{d_i}$ where $\underline{\dim} M = (\dim M_i)_{1 \leq i \leq n} = (d_i)_{1 \leq i \leq n}$ is the dimension vector of M. The same argument gives $\det(g_{-1})_{red} = \prod_{i=1}^n \det(g_-^i)^{d_i}$ and so $s(g \cdot X, M) = \chi(g) \cdot s(X, M)$ where χ is of weight $([M],[M])$. ☐

Remark 2.14. Let X^{-1}, X^0 and M be as before. Consider now $R(X^{-1} \oplus P, X^0 \oplus P)$ for some $0 \neq P \in \operatorname{proj}\Lambda$. Since $\left\langle \left[X \oplus \begin{array}{c} P \\ \shortparallel \\ P \end{array} \right], [M] \right\rangle = \langle [X], [M] \rangle = 0$, the regular function $s(-, M)$ is a semi-invariant for the action of $G(X^{-1} \oplus P, X^0 \oplus P)$ on $R(X^{-1} \oplus P, X^0 \oplus P)$ and satisfies that

$$s\left(\begin{pmatrix} - & 0 \\ 0 & 1_P \end{pmatrix}, -\right) = s(-, M)$$

for any $x \in R(X^{-1}, X^0)$, where $s(-, M)$ is as in Proposition 2.13. Let $x' \in R(X^{-1} \oplus P, X^0 \oplus P)$ and suppose that it belongs to the orbit of the point $\begin{pmatrix} x & 0 \\ 0 & 1_P \end{pmatrix}$ for some $x \in R(X^{-1}, X^0)$. Then, there exists $g \in G(X^{-1} \oplus P, X^0 \oplus P)$ such that

$$s(x', M) = s\left(g \cdot \begin{pmatrix} x & 0 \\ 0 & 1_P \end{pmatrix}\right) = \chi_{([M],[M])}(g) \cdot s\left(\begin{pmatrix} x & 0 \\ 0 & 1_P \end{pmatrix}\right) =$$
$$= \chi_{([M],[M])}(g) \cdot s(x, M)$$

Thus, $s(x', M) \neq 0$ if and only if $s(x, M) \neq 0$, where $s'(-, M)$ is the determinantal semi-invariant associated to M over $R(X^{-1} \oplus P, X^0 \oplus P)$.

We now shift our attention back to $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Remark 2.14 tells us that for any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $M \in \operatorname{mod}\Lambda$ such that $\langle [X], [M] \rangle = 0$, the non-annihilation of the $s(-, M)$ only depends on the isomorphism class of X in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Definition 2.15 (**M-semistability**)**.** Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $M \in \operatorname{mod}\Lambda$. We say that X is *M-semistable* if

i) $\langle [X], [M] \rangle = 0$;

ii) $\exists\ x \in R(X^{-1}, X^0)$ such that $X \simeq \begin{array}{c} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^0 \end{array}$ and $s(x, M) \neq 0$.

2.2.2 Virtual presentation spaces and semi-invariants

Determinantal semi-invariants for projective presentations and their links to cluster algebras where thoroughly studied by K. Igusa, K. Orr, G. Todorov and J. Weyman in [IOTW09, IOTW15]. In their work, they define the ring of *virtual semi-invariants* for any $\theta \in K_0(\operatorname{proj}\Lambda)$ (Definition 2.16), which can be interpreted as those semi-invariants that are well defined for objects in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Notably, they show that when $\Lambda \simeq \Bbbk Q$, where Q is a finite quiver without oriented cycles, then the ring of virtual semi-invariants is spanned by determinantal semi-invariants [IOTW09, Theorem 6.4.1 (Virtual First Fundamental Theorem)]. Although we are interested in the question of whether this holds for a general finite-dimensional algebra Λ, the goal of this text is to find a new categorical significance to this semi-invariants, inspired by all the theory branching off semistability theory in $\operatorname{mod}\Lambda$.

Let $\theta \in K_0(\operatorname{proj}\Lambda) \simeq \mathbb{Z}^n$, and let $PD(\theta) = \{(\eta^{-1},\eta^0) \in \mathbb{Z}^{2n}_{\geq 0} \mid \eta^0 - \eta^{-1} = \theta\}$. Note if we let $\theta^{-1} = -(\min\{0,\theta_i\})_{1\leq i\leq n}$ and $\theta^0 = (\max\{0,\theta_i\})_{1\leq i \leq n}$, then for any $(\eta^{-1},\eta^0) \in PD(\theta)$, there exists $\gamma \in \mathbb{Z}^n_{\geq 0}$ such that $\eta^i = \theta^i + \gamma$ for $i \in \{-1,0\}$. For $\gamma \in \mathbb{Z}^n_{\geq 0}$ we define $P(\gamma) = \bigoplus_{i=1}^n P_i^{\oplus \gamma_i}$. Recall that for any $(\eta^{-1},\eta^0) \in \mathbb{Z}^{2n}_{\geq 0}$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$ we have maps

$$R(P(\eta^{-1}),P(\eta^0)) \longrightarrow R(P(\eta^{-1}) \oplus P(\gamma), P(\eta^0) \oplus P(\gamma))$$

who take any $x \in R(P(\eta^{-1}),P(\eta^0))$ and sends it to $\left(\begin{smallmatrix} x & 0 \\ 0 & 1_{P(\gamma)} \end{smallmatrix}\right)$. We refer to these as *stabilization maps.*

Definition 2.16. [IOTW09] Let $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$. The *virtual presentation space* of θ is the direct limit over $PD(\theta)$

$$R^{vir}(\theta) = \varinjlim_{(\eta^{-1},\eta^0)\in PD(\theta)} R(P(\eta^{-1}),P(\eta^0)).$$

Every $R(\eta^{-1},\eta^0) = R(P(\eta^{-1}),P(\eta^0))$ gives rise to a ring of semi-invariants for the action of the group $G(\eta^{-1},\eta^0) = G(P(\eta^{-1}),P(\eta^0))$. The restriction maps induced by the maps $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1_P \end{smallmatrix}\right)$ described above define an inverse system over $PD(\theta^0 - \theta^{-1})$ of the rings of semi-invariants $SI(R(\theta^{-1},\theta^0))^{G(\theta^{-1},\theta^0)} = \bigoplus_{\bar{d}\in\mathbb{Z}^{2n}}$. The ring of *virtual semi-invariants* for $\theta \in \mathbb{Z}^n$ is the inverse limit over $PD(\theta)$

$$SI^{vir}(\theta) = \varprojlim_{(\eta^{-1},\eta^0)\in PD(\theta)} SI(R(\eta^{-1},\eta^0))^{G(\eta^{-1},\eta^0)}.$$

A *virtual semi-invariant* associated to X is an element f in $SI^{vir}([X])$. The following proposition tell us that, up to adding enough $\underset{P}{\overset{P}{\shortparallel}}$ summands, which does not change the isomorphism class of an object $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, virtual semi-invariants have weights given by $\bar{d} = (d,d)$ for some $d \in \mathbb{Z}^n$.

Proposition 2.17. *[IOTW09, Proposition 3.3.3] Consider $R(X^{-1},X^0)$ where $X^{-1} = \bigoplus_{i=0}^n P_i^{\theta_i^{-1}}$ and $X^0 = \bigoplus_{i=0}^n P_i^{\theta_i^0}$, with its usual $G(X^{-1},X^0)$ action. Suppose that there is a non-zero $f \in SI(R)^{G(X^{-1},X^0),\chi}$ with $\chi = \chi_{\bar{d}}$ for some $\bar{d} = (d^{-1},d^0) \in \mathbb{Z}^{2n}$. Then $d_i^{-1} = d_i^0$, if both θ_i^0 and $\theta_i^{-1} \neq 0$.*

Definition 2.18. Let $f \in SI^{vir}(\theta)$ be a virtual $\chi_{\bar{d}}$-semi-invariant with $\bar{d} = (d^{-1}, d^0)$. By Proposition 2.17 we can suppose that $d^{-1} = d^0 = d$. Indeed, by adding enough summands of the form $\begin{matrix} P \\ \| \\ P \end{matrix}$, we can always assume that $\theta_i^0, \theta_i^{-1} \neq 0$. We say that f has weight $d \in \mathbb{Z}^n$ when this is the case.

Remark 2.19. Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Suppose that there exists a virtual semi-invariant f of weight $d \in K_0(\operatorname{mod}\Lambda)$ such that $f(X) \neq 0$. In particular, there exists $\begin{matrix} X^{-1} \\ \downarrow{\scriptstyle x} \\ X^0 \end{matrix} \simeq X$ such that f defines a $\chi_{(d,d)}$-semi-invariant for the action of $G(X^{-1}, X^0)$ over $R(X^{-1}, X^0)$ with $f(x) \neq 0$, that is, x is $\chi_{(d,d)}$-semistable (see Definition 2.2). This remark inspires the following definition.

Definition 2.20. We say that $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is *virtually d-semistable* for $d \in K_0(\operatorname{mod}\Lambda)$ if there exists a virtual semi-invariant f of weight d such that $f(X) \neq 0$.

Proposition 2.13 implies that the $s(-, M)$ are virtual semi-invariants of weight $\underline{\dim} M$ for those $\theta \in \mathbb{Z}^n$ such that $\langle \theta, [M] \rangle = 0$. Then, if $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is M-semistable for $M \in \operatorname{mod}\Lambda$, it is virtually $[M]$-semistable. Suppose now that $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is virtually d-semistable. By Proposition 2.11, we get that

i) $\langle (-[X^{-1}], [X^0]), (d, d) \rangle = 0$

ii) For any inflation $Y \rightarrowtail X$ in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$, we must have that

$$\langle (-[Y^{-1}], [Y^0]), (d, d) \rangle \geq 0.$$

Recall that for any inflation $Y \rightarrowtail X$ in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ can be lifted to an inflation in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$. The following definition summarizes these facts.

Definition 2.21 (Numerical semistability). Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $d \in \mathcal{K}_0(\operatorname{mod}\Lambda)$. We say that X is *numerically d-semistable* if

1. $\langle [X], d \rangle = 0$,
2. For every inflation $Y \rightarrowtail X$ we have $\langle [Y], d \rangle \geq 0$.

The following theorem links both M-semistability and virtual semistability with numerical semistability for projective presentations.

Proposition 2.22. *Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $d \in \mathbb{Z}^n_{\geq 0}$. If X is virtually d-semistable, then X is numerically d-semistable.*

Proof. Since X is virtually d-semistable, there exists a virtual semi-invariant f of weight d such that $f(x) \neq 0$ for a representative $x \in R(X^{-1}, X^0)$. By Remark 1.28, there exists $P \in \operatorname{proj}\Lambda$ such that $Y \rightarrowtail X \oplus \begin{matrix} P \\ \| \\ P \end{matrix}$ is an inflation in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$. But $X \oplus \begin{matrix} P \\ \| \\ P \end{matrix}$ still satisfies that the virtual semi-invariant f is non-zero. By Proposition 2.11, and noting that $\langle (-[X^{-1}], [X^0]), ([M], [M]) \rangle = \langle [X], [M] \rangle$ we get the result. $\square$

Remark 2.23. We give a proof of Proposition 2.22 that does not rely on geometric arguments. Suppose X is M-semistable and let $X \overset{f}{\rightarrowtail} Y$ be an inflation. Recall that by Remark 1.28, we can find $P \in \operatorname{proj}\Lambda$ and a map f' such that the morphism $Y \xrightarrow{f'} X \oplus \begin{matrix} P \\ \| \\ P \end{matrix}$ is an inflation in $\mathcal{C}^{[-1,0]}\operatorname{proj}\Lambda$ which maps to f when taking homotopies. In particular, we can suppose that the map $Y^{-1} \xrightarrow{f^{-1}} X^{-1}$ is a section, since $X \oplus \begin{matrix} P \\ \| \\ P \end{matrix}$ is M-semistable is X is. By applying $\operatorname{Hom}_\Lambda(-, M)$, we get a commutative square

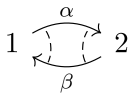
$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(X^0, M) \arrow[r, "f^0"] \arrow[d, "\cong"] & \operatorname{Hom}_\Lambda(Y^0, M) \arrow[d, "\operatorname{Hom}_\Lambda(y,M)"] & \\
\operatorname{Hom}_\Lambda(X^{-1}, M) \arrow[r, two heads, "f^{-1}"] & \operatorname{Hom}_\Lambda(Y^{-1}, M) \arrow[r] & 0.
\end{tikzcd}$$

Since we supposed that f is a section, the map $\operatorname{Hom}_\Lambda(y, M)$ is an epimorphism, and thus

$$\langle [Y], [M] \rangle = \dim \operatorname{Hom}_\Lambda(Y^0, M) - \dim \operatorname{Hom}_\Lambda(Y^{-1}, M) \geq 0.$$

We end this section with two examples. The first shows that numerical semistability does not necessarily imply geometric semistability in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. The second tells us that the subcategory of objects that are numerically d-semistable is not closed under extensions in general.

Example 2.24 (Numerical semistability does not imply geometric semistability). Consider $\Lambda = \mathbb{C}Q/I$, where Q is the quiver with relations



and $I = \langle \alpha\beta, \beta\alpha \rangle$. Consider as well the objects $X_1 = P_1 \xrightarrow{\alpha} P_2$ and $X_2 = P_2 \xrightarrow{\beta} P_1$. We have $\langle [X_1], P_2 \rangle = 0$ and that $\operatorname{Hom}_\Lambda(S_2, P_2) = 0$, and so, the virtual semi-invariant $s(X_1, P_2) = \det(\operatorname{Hom}_\Lambda(P_2, P_2) \xrightarrow{-\circ\alpha} \operatorname{Hom}_\Lambda(P_1, P_2))$ is non-zero, that is, X_1 is P_2-semistable. Likewise, X_2 is P_1-semistable since $s(X_2, P_1) \neq 0$. Consider now $X = X_1 \oplus X_2 \in \operatorname{Hom}_\Lambda(P_1 \oplus P_2, P_1 \oplus P_2)$ with representative $x = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \in R(P_1 \oplus P_2, P_1 \oplus P_2)$. Note that $X = \begin{matrix} P_1 \oplus P_2 \\ \downarrow x \\ P_1 \oplus P_2 \end{matrix}$ satisfies the two properties of Definition 2.21 for the vector $d = [P_1] = [P_2] = (1, 1)$. Since $[X] = (0, 0)$, it satisfies 1). On the other hand, X_1, X_2, $0 \to P_1$ and $0 \to P_2$ are the only possible indecomposable direct summands of objects that are the source of inflations into X, and they all satisfy condition 2). Thus X is numerically $(1, 1)$-semistable. We will show that there is no virtual semi-invariant f of weight $(1, 1)$ such that $f(X) \neq 0$.

Consider

$$x' = \left(\begin{array}{cc|cc} 0 & \mathbf{0}_{1\times n} & 1 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & 1_n & \mathbf{0}_{1\times n} & \mathbf{0}_n \\ \hline 1 & \mathbf{0}_{1\times n} & 0 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & \mathbf{0}_n & \mathbf{0}_{n\times 1} & 1_n \end{array} \right)$$

the image of x by the stabilization map $R(P_1\oplus P_2, P_1\oplus P_2)\to R(P_1^{n+1}\oplus P_2^{n+1}, P_1^{n+1}\oplus P_2^{n+1})$ for $n\gg 0$. Let

$$R_n = R(P_1^{n+1}\oplus P_2^{n+1}, P_1^{n+1}\oplus P_2^{n+1}) = \begin{pmatrix} M_{n+1}(\mathbb{C})\cdot 1_{P_1} & M_{n+1}(\mathbb{C})\cdot\beta \\ M_{n+1}(\mathbb{C})\cdot\alpha & M_{n+1}(\mathbb{C})\cdot 1_{P_2}\end{pmatrix},$$

$$G_n = G(P_1^{n+1}\oplus P_2^{n+1}, P_1^{n+1}\oplus P_2^{n+1}) =$$
$$\begin{pmatrix} GL_{n+1}(\mathbb{C})\cdot 1_{P_1} & M_{n+1}(\mathbb{C})\cdot\beta \\ M_{n+1}(\mathbb{C})\cdot\alpha & GL_{n+1}(\mathbb{C})\cdot 1_{P_2}\end{pmatrix}^{op}\times\begin{pmatrix} GL_{n+1}(\mathbb{C})\cdot 1_{P_1} & M_{n+1}(\mathbb{C})\cdot\beta \\ M_{n+1}(\mathbb{C})\cdot\alpha & GL_{n+1}(\mathbb{C})\cdot 1_{P_2}\end{pmatrix},$$

$$U_n = \begin{pmatrix} 1_n\cdot 1_{P_1} & M_{n+1}(\mathbb{C})\cdot\beta \\ M_{n+1}(\mathbb{C})\cdot\alpha & 1_n\cdot 1_{P_2}\end{pmatrix}^{op}\times\begin{pmatrix} 1_n\cdot 1_{P_1} & M_{n+1}(\mathbb{C})\cdot\beta \\ M_{n+1}(\mathbb{C})\cdot\alpha & 1_n\cdot 1_{P_2}\end{pmatrix},$$

$$(G_n)_{red} =$$
$$\begin{pmatrix} GL_{n+1}(\mathbb{C})\cdot 1_{P_1} & 0 \\ 0 & GL_{n+1}(\mathbb{C})\cdot 1_{P_2}\end{pmatrix}^{op}\times\begin{pmatrix} GL_{n+1}(\mathbb{C})\cdot 1_{P_1} & 0 \\ 0 & GL_{n+1}(\mathbb{C})\cdot 1_{P_2}\end{pmatrix},$$

where U_n is the unipotent radical of $G_n = (G_n)_{red}\rtimes U_n$. The group G_n acts on R_n in the following way. For every $g = \begin{pmatrix} X & Y \\ Z & W\end{pmatrix}\times\begin{pmatrix} X' & Y' \\ Z' & W'\end{pmatrix}\in G_n$ and $y = \begin{pmatrix} A & B \\ C & D\end{pmatrix}\in R_n$ we have that

$$g\cdot y = \begin{pmatrix} X'AX & X'AY + X'BW + Y'DW \\ Z'AX + W'CX + W'DZ & W'DW\end{pmatrix}.$$

Any G_n-semi-invariant f on R_n must be a U_n-invariant function. In particular, f must be invariant for the action of the subgroup $V = 1_{2n+2}\times\begin{pmatrix} 1_{n+1} & M_{n+1}(\mathbb{C})\cdot\beta \\ 0 & 1_{n+1}\end{pmatrix}$. Given that

$$\begin{pmatrix} 1_{n+1} & Y' \\ 0 & 1_{n+1}\end{pmatrix}\cdot\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{pmatrix} A & B+Y'D \\ C & D\end{pmatrix},$$

then the ring of invariants $\mathbb{C}[R_n]^V = \mathbb{C}[A,B,C,D]^V$ is isomorphic to $\mathbb{C}[A,C]\otimes\mathbb{C}[B,D]^{V'}$, where $\mathbb{C}[B,D]^{V'}$ is the ring of invariants of the action of $V' = \begin{pmatrix} 1_{n+1} & M_{n+1}(\mathbb{C}) \\ 0 & 1_{n+1}\end{pmatrix}$ over the set of $2n+2$ by $n+1$ matrices $\begin{pmatrix} B \\ D\end{pmatrix}$ given by left multiplication. In [Pom87, Section 4.2], the author explicity describes a basis of the invariant functions on the variety of $N\times M$ matrices under the action of certain unipotent subgroups of $GL_N(\mathbb{C})$ by left multiplication. Applying these results to our particular case, we obtain that $\mathbb{C}[B,D]^{V'} = \mathbb{C}[D]$, and thus $\mathbb{C}[R_n]^V = \mathbb{C}[A,C,D]$. A similar argument shows that the ring of invariants on R_n by the action of $H = 1_{2n+2}\times\begin{pmatrix} 1_{n+1} & 0 \\ M_{n+1}(\mathbb{C})\cdot\alpha & 1_{n+1}\end{pmatrix}$ is $\mathbb{C}[A,B,D]$, which in turn implies that $\mathbb{C}(R_n)^{V\cdot H} = \mathbb{C}[A,D]$, where $V\cdot H = 1_{2n+2}\times\begin{pmatrix} 1_{n+1} & M_{n+1}(\mathbb{C})\cdot\beta \\ M_{n+1}(\mathbb{C})\cdot\alpha & 1_{n+1}\end{pmatrix}$. Moreover, the functions in $\mathbb{C}[A,D]$ are also invariant for the action of

$$\begin{pmatrix} 1_{n+1} & M_{n+1}(\mathbb{C})\cdot\beta \\ M_{n+1}(\mathbb{C})\cdot\alpha & 1_{n+1}\end{pmatrix}^{op}\times 1_{2n+2},$$

which gives us that $\mathbb{C}[R_n]^{U_n} = \mathbb{C}[A, D]$. The only $(G_n)_{red}$-semi-invariants of weight $(1, 1)$ on $\mathbb{C}[A, D]$ are the functions $f_k(A, D) = k \cdot \det(A) \cdot \det(D)$ for $k \in \mathbb{C}^*$ [DK15, Theorem 4.4.4], and for all of them, $f_k(x') = 0$, since $A_{x'} = D_{x'}$ are not of full rank. That is, X is not geometrically $(1, 1)$-semistable.

Example 2.25 (**Numerical semistability is not closed under extensions**). Consider again the quiver of the last example. As we have seen, both $X_1 = P_1 \xrightarrow{\alpha} P_2$ and $X_2 = P_2 \xrightarrow{\beta} P_1$ are numerically $(1, 1)$-semistable. In $\mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$, the following sequence is a conflation

$$\begin{tikzcd}
P_1 \arrow[r, equal] \arrow[d, "\alpha"] & P_1 \arrow[r, "\alpha"] \arrow[d, "0"] & P_2 \arrow[d, "\beta"] \\
P_2 \arrow[r, "\beta"] & P_1 \arrow[r, equal] & P_1
\end{tikzcd}$$

However, $P_1 \oplus P_1[1]$ is not $(1, 1)$-numerically semistable. Indeed $\langle -[P_1], (1, 1) \rangle = -1 \leq 0$.

Chapter 3

Thick subcategories of projective presentations

In this chapter we introduce the study of thick subcategories of the extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ in the context of τ-tilting theory. The results of this chapter are taken from [Gar23, Sections 2 and 3].

3.1 Structures in $\operatorname{mod}\Lambda$

In their seminal paper on τ-tilting theory [AIR14], T. Adachi, O. Iyama and I. Reiten studied the relationship between several classes of objects, namely, support τ-tilting modules, 2-term silting complexes, and functorially finite torsion classes. Since then, driven by applications to cluster theory [BMR$^+$06, Ami09] and stability conditions [Asa20, BST22], among others, many classes of objects have been added to this list. In the category $\operatorname{mod}\Lambda$ of finitely generated modules over an associative algebra Λ, this list includes

- support τ-tilting pairs [IT09, AIR14, DIR$^+$23],
- the lattice of torsion pairs [Dic66, IT09, IRTT15],
- the poset of wide subcategories [Hov01, BM21, BH23],

to name a few. In this section, we recall how these classes of objects relate to each other. The main theorem is the following :

Theorem 3.1. *[IT09, Theorem 1.1][AIR14, Theorem 0.5][MŠ17, Theorem 30][Yur18, Theorem 1.2] [BST19, Theorem 1.1] Let Λ be a finite-dimensional $\Bbbk$-algebra. There are explicit bijections between the sets of*

1) Isomorphism classes of basic support τ-tilting modules in $\operatorname{mod}\Lambda$.

2) Functorially finite torsion pairs in $\operatorname{mod}\Lambda$.

3) Left finite wide subcategories of $\operatorname{mod}\Lambda$.

4) Left finite semistable subcategories of $\operatorname{mod}\Lambda$.

We will particularly focus on the case where Λ is a hereditary algebra. This emphasis will serve as guidance for translating Theorem 3.1 into the hereditary extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, as we will explore in Section 3.2.

3.1.1 Torsion classes, wide subcategories and support τ-tilting pairs

Let Λ be a finite-dimensional $\mathbb{k}$-algebra and let $n = |\Lambda|$ be the number of non isomorphic indecomposable diret summands of Λ. Frequently, Λ is the path algebra $\mathbb{k}Q/I$ of a quiver Q with n vertices modulo some admissible relations. Let $\mathcal{C} \subset \operatorname{mod}\Lambda$, we will denote

$$\mathcal{C}^{\perp} = \{M \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_{\Lambda}(\mathcal{C}, M) = 0\}$$
$$^{\perp}\mathcal{C} = \{M \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_{\Lambda}(M, \mathcal{C}) = 0\},$$

The *Hom-orthogonal* subcategories associated to $\mathcal{C}$. We define as well

$$\operatorname{Fac}(\mathcal{C}) = \{N \in \operatorname{mod}\Lambda \mid \exists\ M \to N \to 0 \text{ exact sequence with } M \in \operatorname{add}(\mathcal{C})\}$$
$$\operatorname{Filt}(\mathcal{C}) = \bigcap_{\substack{\mathcal{D} \subset \operatorname{mod}\Lambda \\ \mathcal{D} \text{ closed under} \\ \text{extensions} \\ containing \mathcal{C}}} \mathcal{D},$$

in other words, $\operatorname{Fac}(\mathcal{C})$ is the full subcategory of $\operatorname{mod}\Lambda$ which contains all quotients of of finite sums of objects in $\mathcal{C}$ and $\operatorname{Filt}(\mathcal{C})$ is the smallest full subcategory of $\operatorname{mod}\Lambda$ containing $\mathcal{C}$ that is closed under extensions. In particular $\operatorname{Filt}(\mathcal{C})$ is the subcategory of all modules M admitting a filtration

$$0 = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_l = M$$

such that $C_i/C_{i-1} \in \mathcal{C}$ for all $1 \leq i \leq l$.

Recall that we have dualities

$$D := \operatorname{Hom}_{\mathbb{k}}(-, \mathbb{k}) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op} \quad \text{and} \quad (-)^* := \operatorname{Hom}_{\Lambda}(-, \Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$$

which induce an equivalence $\nu := D(-)^* : \operatorname{proj}\Lambda \to \operatorname{inj}\Lambda$ known as the *Nakayama functor*. For any module $M \in \operatorname{mod}\Lambda$, its Auslander-Reiten translation τM is given by

$$\tau M = H^{-1}(\nu X_M) = H^0(X_{H^{-1}(\nu X_M)}) = H^0(\tau X_M),$$

where X_M and $X_{H^{-1}(\nu X_M)}$ are the minimal projective presentations of M and $H^{-1}(\nu X_M)$ respectively (see Section 1.3.2).

The following definition is due to by T. Adachi, O. Iyama and I. Reiten :

Definition 3.2. [AIR14, Definition 0.3] Let M and P be finite-dimensional Λ-modules such that P is projective. We say that (M, P) is a support τ-rigid pair if and only if

1. $\operatorname{Hom}_{\Lambda}(P, M) = 0$,
2. $\operatorname{Hom}_{\Lambda}(M, \tau M) = 0$.

We say that (M, P) is support τ-tilting if it is rigid and satisfies that $|M| + |P| = |\Lambda|$. We denote by $\mathrm{s}\tau\text{-rigid}\,\Lambda$ the set of basic support τ-rigid pairs up to isomorphism, and by $\mathrm{s}\tau\text{-tilt}\,\Lambda \subset \mathrm{s}\tau\text{-rigid}\,\Lambda$ the subset of those pairs that are τ-tilting.

Support τ-rigid (resp. τ-tilting) pairs are a generalization of partially tilting (resp. tilting) modules inspired by the categorification of cluster algebras. Similar to classical partially tilting modules, every support τ-rigid pair corresponds to a *torsion pair.*

Definition 3.3. Let $\mathcal{T}$ and $\mathcal{F}$ be full additive subcategories of $\operatorname{mod}\Lambda$. We say that $(\mathcal{T},\mathcal{F})$ is a *torsion pair* if

i) $\operatorname{Hom}_\Lambda(\mathcal{T},\mathcal{F}) = 0$,

ii) For any $M \in \operatorname{mod}\Lambda$ there exists $t_M \in \mathcal{T}$, $f_M \in \mathcal{F}$ and a short exact sequence

$$0 \to t_M \to M \to f_M \to 0.$$

Remark 3.4. If $(\mathcal{T},\mathcal{F})$ is a torsion pair, the exact sequence in Definition 3.3 ii) is unique up to isomorphism.

Proposition 3.5. *[ASS06, Propositions 1.4 and 1.5] Let $\mathcal{F}$ and $\mathcal{T}$ be full subcategories of* $\operatorname{mod}\Lambda$. *The following are equivalent:*

i) $(\mathcal{T},\mathcal{F})$ is a torsion pair.

ii) $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = {}^\perp\mathcal{F}$.

iii) $\mathcal{T}$ is closed under extensions, quotients and $\mathcal{F} = \mathcal{T}^\perp$.

iv) $\mathcal{F}$ is closed under extensions, submodules and $\mathcal{T} = {}^\perp\mathcal{F}$.

Proof. The implications i) $\Longrightarrow$ ii) $\Longrightarrow$ iii) follow from the definitions and the right-exacteness of the functor $\operatorname{Hom}_\Lambda(-, M)$ for all $M \in \operatorname{mod}\Lambda$. To prove iii) $\Longrightarrow$ i), take $M \in \operatorname{mod}\Lambda$ and let t_M the largest submodule of M satisfying that $t_M \in \mathcal{T}$ (it exists since the set of submodules of M in $\mathcal{T}$ is stable under sums). Let $N \in \mathcal{T}$ and $f \in \operatorname{Hom}_\Lambda(N, M/t_M)$. Since $\mathcal{T}$ is closed under quotients, $\operatorname{Im} f \in \mathcal{T}$ and there exists $M' \subset M$ such that $t_M \subset M'$ and $M'/t_M \simeq \operatorname{Im} f$. Using that $\mathcal{T}$ is closed under extensions, we get that $M' \in \mathcal{T}$ and $M' = t_M$. Hence, $f = 0$ and $M/t_M \in \mathcal{T}^\perp = \mathcal{F}$. □

Remark 3.6. Proposition 3.5 says that in order to describe a torsion pair $(\mathcal{T},\mathcal{F})$, it is sufficient to know either its associated *torsion class* $\mathcal{T}$ or its *torsion-free class* $\mathcal{F}$, since they determinate each other.

Definition 3.7. Let $\mathcal{C}$ be a full subcategory of $\operatorname{mod}\Lambda$. We say that $M \in \mathcal{C}$ is *Ext-projective* if $\operatorname{Ext}^1_\Lambda(M,\mathcal{C}) = 0$. We will denote by $P(\mathcal{C})$ the direct sum of all indecomposable Ext-projective modules in $\mathcal{C}$. We will say that M is *split projective* in $\mathcal{C}$ if for every epimorphism $C \xrightarrow{f} M \to 0$ with $C \in \mathcal{C}$, then f splits.

We say that a torsion pair $(\mathcal{T},\mathcal{F})$ is *functorially finite* if every Λ-module admits both a right and a left $\mathcal{T}$-approximation. We denote by $\text{f-tors}\,\Lambda \subset \operatorname{tors}\Lambda$ the subset of functorially finite torsion classes in $\operatorname{mod}\Lambda$. Classical examples of functorially finite torsion classes are the subcategories $\operatorname{Fac}(T)$ for a tilting module T. One of the many features of τ-tilting theory is that it provided a characterization of all functorially finite torsion classes for any finite-dimensional algebra Λ.

Proposition 3.8. *[AIR14, Proposition 1.2] For any $M, N \in \operatorname{mod}\Lambda$, then :*

1. $\operatorname{Hom}_\Lambda(M, \tau N) = 0$ *if and only if* $\operatorname{Ext}^1_\Lambda(N, \operatorname{Fac}(M)) = 0$
2. *If M is τ-rigid, then* $\operatorname{Fac}(M)$ *is a functorially finite torsion class and* $M \in P(\operatorname{Fac}(M))$.

Theorem 3.9. *[AIR14, Theorem 2.7] There is a bijection between the set* $\mathrm{s}\tau\text{-tilt}\,\Lambda$ *of isomorphism classes of support τ-tilting pairs and the set of functorially finite torsion classes* $\text{f-tors}\,\Lambda$ *given by*

$$\begin{aligned} \mathrm{s}\tau\text{-tilt}\,\Lambda &\longleftrightarrow \text{f-tors}\,\Lambda \\ (M,P) &\longmapsto \mathrm{Fac}(M) \\ (P(\mathcal{T}),Q) &\longleftarrow\!\!\shortmid\ \mathcal{T} \end{aligned}$$

where Q is the maximal basic projective module satisfying $\mathrm{Hom}_\Lambda(Q,\mathcal{T})=0$.

Remark 3.10. Let $\mathcal{T}\in\text{f-tors}\,\Lambda$, then there exists $(M,P)\in\mathrm{s}\tau\text{-tilt}\,\Lambda$ such that $\mathcal{T}=\mathrm{Fac}(M)$. In particular, $\mathcal{T}$ is covariantly finite and thus Λ admits a left $\mathcal{T}$-approximation $\Lambda\xrightarrow{f}M_0$. Since $\mathcal{T}$ is closed under quotients and $M_0\in\mathcal{T}$, we get an exact sequence

$$\Lambda\xrightarrow{f}M_0\to M_1\to 0 \tag{3.1.1}$$

where $M_1\simeq M_0/\operatorname{Im}f_0\in\mathcal{T}$. Moreover, $\mathrm{add}(M_0\oplus M_1)=\mathrm{add}(M)$ (see for instance [AIR14, Lemma 2.20] and [AI12, Proposition 2.4]). This gives a canonical decomposition of M as $M_\rho\oplus M_\lambda$ where M_ρ and M_λ are the basic τ-rigid modules satisfying $\mathrm{add}(M_\rho)=\mathrm{add}(M_0)$, $\mathrm{add}(M_\lambda)=\mathrm{add}(M_1)$ and $\mathcal{T}=\mathrm{Fac}(M_0)$. The following lemma provides a characterization of this decomposition.

Lemma 3.11. *[MŠ17, Lemma 3.7][BH23, Lemma 2.6] Let (M,P) be a support τ-tilting pair and let $\mathcal{T}=\mathrm{Fac}(M)$, then there is a decomposition*

$$M=M_\rho\oplus M_\lambda$$

such that $\mathcal{T}=\mathrm{Fac}(M_\rho)$, M_ρ is split projective in $\mathcal{T}$ and no direct summand of M_λ is split projective in $\mathcal{T}$. Moreover, if there exists an exact sequence

$$\Lambda\xrightarrow{f}M_0\to M_1\to 0$$

where f is a minimal left $\mathcal{T}$-approximation of Λ, then $\mathrm{add}(M_0)=\mathrm{add}(M_\rho)$ *and* $\mathrm{add}(M_1)=\mathrm{add}(M_\lambda)$.

Besides their importance in the study of the category $\operatorname{mod}\Lambda$, torsion classes have deep combinatorial implications. The set $\operatorname{tors}\Lambda$ equipped with the order given by inclusion is a *complete lattice*, that is, a poset such any subset of elements in have both an infimum (or *meet*) and a supremum (or *join*). In $\operatorname{tors}\Lambda$, these operations are given by arbitrary intersection, and $\mathrm{Filt}(\mathrm{Fac}(-))$ respectively. The study of $\operatorname{tors}\Lambda$ has become part of the tool kit of mathematicians to study lattices [IT09, IRRT18, RST21, BDH23]. One of the first results in this direction was introduced by C. Ingalls and H. Thomas, who, using techniques involving torsion pairs, showed that there exists a bijection between *non-crossing partitions* of extended Dynkin quiver Q and clusters of the acyclic algebra associated to Q. A crucial step in their proof is the construction of a bijection between functorially finite torsion classes and a certain class of *wide subcategories*.

Definition 3.12. Let $\mathcal{W}\subset\operatorname{mod}\Lambda$ be a full subcategory of $\operatorname{mod}\Lambda$. We say that $\mathcal{W}$ is *wide* if it is additive and closed under extensions, kernels and cokernels. That is, $\mathcal{W}$ is wide if and only is it is an abelian subcategory of $\operatorname{mod}\Lambda$. We denote by $\operatorname{wide}\Lambda$ the set of wide subcategories of $\operatorname{mod}\Lambda$. For a subcategory $\mathcal{H}\subset\operatorname{mod}\Lambda$, we denote by $\mathrm{wide}(\mathcal{H})$ the smallest subcategory containing $\mathcal{H}$.

Proposition 3.13. *[IT09, Proposition 2.12] Let Λ be any finite-dimensional algebra. There exists a well defined map*

$$\operatorname{tors}\Lambda \xrightarrow{\alpha} \operatorname{wide}\Lambda$$

given by $\alpha(\mathcal{T}) = \{M \in \mathcal{T} \mid \forall\ N \xrightarrow{f} M$ *such that* $N \in \mathcal{T},\ \operatorname{Ker} f \in \mathcal{T}\}$.

Proof. Let $M, N \in \alpha(\mathcal{T})$ and let $f : N \to M$ be a map. We will show that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are in $\alpha(\mathcal{T})$. Let $g : L \to \operatorname{Ker} f$ for some $L \in \mathcal{T}$. Then the canonical inclusion $\iota : \operatorname{Ker} f \to N$ induces a map $\iota g : L \to N$. Since $N \in \alpha(\mathcal{T})$, $\operatorname{Ker} \iota g = \operatorname{Ker} g \in \mathcal{T}$. Now let $g : L \to \operatorname{Coker} f$, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\pi^* \arrow[r] & H \arrow[r, "\pi^*"] \arrow[d, "g^*"] & L \arrow[r] \arrow[d, "g"] & 0 \\
 & N \arrow[r, "f"] & M \arrow[r, "\pi"] & \operatorname{Coker} f \arrow[r] & 0
\end{tikzcd}$$

where $\pi : M \to \operatorname{Coker} f$ is the canonical projection and (H, π^*, g^*) is the pullback of π and g. Since $\mathcal{T}$ is closed under extensions and $\operatorname{Ker}\pi^* \simeq \operatorname{Ker}\pi \simeq \operatorname{Im} f \in \mathcal{T}$, we have that $H \in \mathcal{T}$. Now since $M \in \alpha(\mathcal{T})$, $\operatorname{Ker} g^* \simeq \operatorname{Ker} g \in \mathcal{T}$.

We have left to prove that $\alpha(\mathcal{T})$ is closed under extensions. Consider a short exact sequence $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$ such that $L, N \in \alpha(\mathcal{T})$. Let $g : P \to M$ for $P \in \mathcal{T}$, then we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} g \arrow[r] \arrow[dd] & \operatorname{Ker}\pi g \arrow[d] \arrow[ddl, "\psi"'] & & \\
 & & P \arrow[d, "g"] & & \\
0 \arrow[r] & L \arrow[r, "\iota"] & M \arrow[r, "\pi"] & N \arrow[r] & 0.
\end{tikzcd}$$

Since $N, L \in \alpha(\mathcal{T})$, we have that $\operatorname{Ker}\pi g \in \mathcal{T}$ and thus $\operatorname{Ker}\psi \simeq \operatorname{Ker}\iota\psi \simeq \operatorname{Ker} g \in \mathcal{T}$. □

One could think of the map α in Proposition 3.13 as the operation of taking a torsion class $\mathcal{T}$, which is already closed under extensions and some cokernels, and forcing the closure under kernels, in order to make it into a wide subcategory. A reciprocal map for α should then take a wide subcategory $\mathcal{W}$, which is already closed under extensions and some quotients, and send it to a category where we force closure under all quotients.

Proposition 3.14. *[IT09, Proposition 2.13] Let Λ be a finite-dimensional hereditary algebra. There exists a well defined map*

$$\begin{aligned} \operatorname{wide}\Lambda &\longrightarrow \operatorname{tors}\Lambda \\ \mathcal{W} &\longmapsto \operatorname{Fac}(\mathcal{W}). \end{aligned}$$

Moreover, $\operatorname{Fac}(\mathcal{W})$ *is the smallest torsion class containing* $\mathcal{W}$.

Proof. Let $\mathcal{W}$ be a wide subcategory. By definition $\operatorname{Fac}(\mathcal{W})$ is closed under quotients. Consider a short exact sequence $0 \to L \to M \to N \to 0$ and epimorphisms $f : \bar{L} \to L$ and $g : \bar{N} \to N$ with $\bar{N}, \bar{L} \in \mathcal{W}$. Since Λ is hereditary, then we have an exact sequence $\operatorname{Ext}^1_\Lambda(N,\bar{L}) \xrightarrow{f_*} \operatorname{Ext}^1_\Lambda(N,L) \to 0$, which implies that we have an extension $0 \to \bar{L} \to X \to N \to 0$ which fits in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bar{L} \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near end] & M' \arrow[r, "p"] \arrow[d, "f_*"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd} \tag{3.1.2}$$

where the first square is a pushout. By taking the pullback of g and p, we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \bar{L} \arrow[r] \arrow[d, equal] & \bar{M} \arrow[r, "p^*"] \arrow[d, "g^*"] \arrow[dr, phantom, "\ulcorner", very near start] & \bar{N} \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \bar{L} \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near end] & M' \arrow[r, "p"] \arrow[d, "f_*"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

with exact rows. In particular, $\bar{M} \in \mathcal{W}$, since $\mathcal{W}$ is closed under extensions. Given that f and g are epimorphisms, it follows that $f_*g^* : \bar{M} \to M$ is also one, and thus $M \in \operatorname{Fac}(\mathcal{W})$. ☐

Note that in the previous proposition it is sufficient to suppose that $\mathcal{W}$ is closed under extensions. However, the hypothesis of Λ being *hereditary* is essential: without it the existence of the object M' appearing in 3.1.2 is not guaranteed. It turns out that for a non-hereditary algebra Λ, $\operatorname{Fac}(\mathcal{W})$ will not be closed under extensions in general. This issue was fixed by F. Marks and J. Šťovíček in [MŠ17], who proved that the right torsion class containing $\mathcal{W}$ that should be looked at is $\operatorname{Filt}(\operatorname{Fac}(\mathcal{W}))$.

Proposition 3.15. *[MŠ17, Lemma 3.1] Let Λ be any finite-dimensional algebra. For any wide subcategory $\mathcal{W} \subset \operatorname{mod}\Lambda$, $\operatorname{Filt}(\operatorname{Fac}(\mathcal{W}))$ is a torsion class.*

Proposition 3.16. *[MŠ17, Proposition 3.3] [IT09, Proposition 2.13] If $\mathcal{W}$ is a wide subcategory, then $\mathcal{W} = \alpha(\operatorname{Filt}(\operatorname{Fac}(\mathcal{W})))$. In particular, the map*

$$\begin{aligned} \operatorname{wide}\Lambda &\longrightarrow \operatorname{tors}\Lambda \\ \mathcal{W} &\longmapsto \operatorname{Filt}(\operatorname{Fac}(\mathcal{W})) \end{aligned}$$

yields an injection from $\operatorname{wide}\Lambda$ to $\operatorname{tors}\Lambda$.

Proof. We will show that $\alpha(\operatorname{Fac}(\mathcal{W})) = \mathcal{W}$ for any wide subcategory $\mathcal{W}$ following the original argument by Ingalls and Thomas. When Λ is hereditary, this equality together with Proposition 3.14 gives the result. For the general case see [MŠ17].

We will first prove that $\mathcal{W} \subset \alpha(\operatorname{Fac}(\mathcal{W}))$. For this let $M \in \mathcal{W}$, and let $f : N \to M$ and $\pi : \bar{N} \to N$ be maps with $\bar{N} \in \mathcal{W}$ and such that f is surjective. We then get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\pi \arrow[r] \arrow[d, equal] & \operatorname{Ker} f\pi \arrow[r] \arrow[d] & \operatorname{Ker} f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\pi \arrow[r] & \bar{N} \arrow[r, "\pi"] \arrow[d, equal] & N \arrow[r] \arrow[d, "f"] & 0 \\
& & \bar{N} \arrow[r, "f\pi"] & M. &
\end{tikzcd}$$

Since both $\bar{N}$ and M lie in $\mathcal{W}$, we have that $\operatorname{Ker} f\pi \in \mathcal{W}$ and thus $\operatorname{Ker} f \in \operatorname{Fac}(\mathcal{W})$. We get that $M \in \alpha(\operatorname{Fac}(\mathcal{W}))$ as wished.

Suppose now that $M \in \alpha(\operatorname{Fac}(\mathcal{W}))$. Since $M \in \operatorname{Fac}(\mathcal{W})$, there exists $\bar{M} \in \mathcal{W}$ and an epimorphism $\pi : \bar{M} \to M$. Let $\rho : K \to \operatorname{Ker} \pi$ be an epimorphism with $K \in \mathcal{W}$, which exists since $\operatorname{Ker} \pi \in \operatorname{Fac}(\mathcal{W})$ since $M \in \alpha(\operatorname{Fac}(\mathcal{W}))$. Let $\iota : \operatorname{Ker} \pi \to \bar{M}$ the natural inclusion. Then the map $\iota\rho : K \to \bar{W}$ satisfies that $M \simeq \operatorname{Coker} \iota \simeq \operatorname{Coker} \iota\rho$. This turn implies that $M \in \mathcal{W}$, since $\mathcal{W}$ is wide. □

One of the most consequential results in [IT09] and [MŠ17] is the fact that maps α (Proposition 3.13) and $\operatorname{Filt}(\operatorname{Fac}(-))$ are inverse of each other when restricted to the set $\operatorname{f-tors}\Lambda$ of functorially finite torsion classes and $\alpha(\operatorname{f-tors}\Lambda)$. In both versions of the proof (hereditary and non-hereditary), a key fact is the description of $\alpha(\mathcal{T})$ when $\mathcal{T} = \operatorname{Fac}(M)$ for a support τ-tilting pair (M, P) in terms of its canonical decomposition $M = M_\rho \oplus M_\lambda$ introduced in Lemma 3.11 :

Proposition 3.17. *[MŠ17, Lemma 3.8] Let $\mathcal{T} = \operatorname{Fac}(M)$ be a functorially finite torsion class in $\operatorname{mod}\Lambda$ such that $(M, P) \in \operatorname{s\tau-tilt}\Lambda$. Then*

$$\alpha(\mathcal{T}) = \operatorname{Fac}(M_\rho) \cap M_\lambda^{\perp},$$

where M_ρ and M_λ are as in Lemma 3.11.

Proof. First note that $\operatorname{Fac}(M_\rho) = \mathcal{T}$ by Lemma 3.11. Let $L \in \mathcal{T} \cap M_\lambda^{\perp}$ and take a map $g : N \to L$ with $N \in \mathcal{T}$. Since $M_\lambda^{\perp}$ is closed under quotients, we have that $\operatorname{Im} g \in \mathcal{T} \cap M_\lambda^{\perp}$. Let $\pi : \Lambda^n \to \operatorname{Ker} f$ be an epimorphism and consider the commutative diagram

$$\begin{tikzcd}
 & \Lambda^n \arrow[r, "f^n"] \arrow[d, "\pi"] & M_0^n \arrow[r] \arrow[d, "\bar{\pi}"] & M_1^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker} g \arrow[r, "\iota"] & N \arrow[r, "g"] & L \arrow[r] & 0
\end{tikzcd}$$

where the first row is induced by the exact sequence $\Lambda \xrightarrow{f} M_0 \to M_1 \to 0$ introduced in Lemma 3.11. Since $\operatorname{Hom}_\Lambda(M_1, L) = 0$, there exists a map $\pi' : M_0^n \to \operatorname{Ker} g$ such that $\iota\pi' = \bar{\pi}$ and

$$\iota\pi' f^n = \bar{\pi} f^n = \iota\pi.$$

Given that ι is a monomorphism, $\pi' f^n = \pi$ and π' is an epimorphisms since π is. We conclude that $\operatorname{Ker} g \in \operatorname{Fac}(M_\rho) = \operatorname{Fac}(M)$.

Now let $L \in \alpha(\mathcal{T}) = \alpha(\operatorname{Fac}(M))$ and $h : M_1 \to L$. Remark that since $\operatorname{Im} h \subset L \in \alpha(\mathcal{T})$, we have that $\operatorname{Im} h \in \alpha(\mathcal{T})$. Consider the commutative diagram of exact rows and columns

$$\begin{tikzcd}
 & \Lambda \arrow[r, equal] \arrow[d, "\bar{f}"] & \Lambda \arrow[d, "f"] & & \\
0 \arrow[r] & \operatorname{Ker} \bar{h} \arrow[r, "\rho"] \arrow[d] & M_0 \arrow[r, "\bar{h}"] \arrow[d] & \operatorname{Im} h \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker} h \arrow[r] \arrow[d] & M_1 \arrow[r, "h"] \arrow[d] & \operatorname{Im} h \arrow[r] & 0 \\
 & 0 & 0 & & .
\end{tikzcd}$$

Given that $M_0 \in \mathcal{T}$ and $\operatorname{Im} h \in \alpha(\mathcal{T})$, we get that $\operatorname{Ker} \bar{h} \in \mathcal{T}$. But f is a minimal left $\mathcal{T}$-approximation of Λ, so there must exist a map $l : M_0 \to \operatorname{Ker} \bar{h}$ such that $lf = \bar{f}$. In particular, $f = \rho\bar{f} = \rho l f$. Since f is minimal, ρl is an isomorphism and thus $\operatorname{Im} h = 0$. We conclude that $\operatorname{Hom}_\Lambda(M_1, L) = 0$, and since $\operatorname{add}(M_1) = \operatorname{add}(M_\lambda)$ we get that $L \in M_\lambda^\perp$. $\square$

Remark 3.18. In the case where Λ is an hereditary algebra, Ingalls and Thomas showed in [IT09, Proposition 2.15] that if (M, P) is a support τ-tilting pair, then

$$\begin{aligned}\alpha_s(\operatorname{Fac}(M)) := \{L \in \operatorname{Fac}(M) \,|\, \forall \text{ epimorphism } N \xrightarrow{f} L \textit{ with } N \in \operatorname{add}(M_\rho),\ \operatorname{Ker} f \in \mathcal{T}\} \\ = \alpha(\operatorname{Fac}(M)).\end{aligned}$$

Since M_ρ is split projective in $\mathcal{T}$, then it clearly belongs to $\alpha_s(\operatorname{Fac}(M))$ and by Proposition 3.17 $M_\rho \in M_\lambda^\perp$. Moreover, since $\operatorname{Fac}(M) = \operatorname{Fac}(M_\rho)$, for every $L \in \alpha(\operatorname{Fac}(M))$ there exists a short exact sequence $0 \to \operatorname{Ker} \pi \xrightarrow{\iota} M' \xrightarrow{\pi} L \to 0$ where $M' \in \operatorname{add}(M_\rho)$ and thus $\operatorname{Ker} \pi \in \operatorname{Fac}(M)$. Now since Λ is hereditary, we have a surjection $\iota^* : \operatorname{Ext}^1_\Lambda(M', X) \to \operatorname{Ext}^1_\Lambda(\operatorname{Ker} \pi, X)$ for every $X \in \operatorname{Fac}(M)$. Given that M' is ext-projective in $\operatorname{Fac}(M)$, then so is $\operatorname{Ker} \pi$ and $\operatorname{Ker} \pi \in \operatorname{add}(M)$. Moreover, given that $M_\rho \in M_\lambda^\perp$ and that ι is a monomorphism, no direct summand of $\operatorname{Ker} \pi$ is in $\operatorname{add}(M_\lambda)$. We conclude that for all $L \in \alpha(\operatorname{Fac}(M))$, there exists a short exact sequence

$$0 \to \bar{M} \to M' \to L \to 0$$

such that $\bar{M}, M' \in \operatorname{add}(M_\rho)$.

Reciprocally, let $L \in \operatorname{Fac}(M)$ be such that such a short exact sequence exists and let $f : N \to L$ be an epimorphism with $N \in \operatorname{add}(M_\rho)$ and $\iota : \operatorname{Ker} f \to N$ the natural inclusion. Since N is ext-projective in $\operatorname{Fac}(M)$, $\operatorname{Ext}^1_\Lambda(N, \bar{M}) = 0$. We get that the connection map $\delta(\iota) : \operatorname{Hom}_\Lambda(\operatorname{Ker} f, \bar{M}) \to \operatorname{Ext}^1_\Lambda(L, \bar{M})$ is surjective, and hence there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} f \arrow[r, "\iota"] \arrow[d] & N \arrow[r, "f"] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bar{M} \arrow[r] & M' \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

such that first commutative square is a pushout. In particular, there exists a short exact sequence

$$0 \to \operatorname{Ker} f \to N \oplus \bar{M} \to M' \to 0$$

where $N \oplus M' \in \operatorname{Fac}(M)$. But M' is ext-projective in $\operatorname{Fac}(M)$, so the sequence splits and $\operatorname{Ker} f \in \operatorname{Fac}(M)$. That is, $L \in \alpha_s(\operatorname{Fac}(M)) = \alpha(\operatorname{Fac}(M))$. We conclude that when Λ is hereditary,

$$\alpha(\operatorname{Fac}(M)) = \{L \in \operatorname{Fac}(M) \mid \exists\ \bar{M}, M' \in \operatorname{add}(M_\rho) \text{ and } 0 \to \bar{M} \to M' \to L \to 0\}.$$

In particular, $\alpha(\operatorname{Fac}(M))$ is an hereditary abelian category with enough projective objects, which are given by $\operatorname{add}(M_\rho)$.

Proposition 3.19. *[MŠ17, Propostion 3.9] [IT09, Proposition 2.16] Let* $\mathcal{T} = \operatorname{Fac}(M)$ *be a functorially finite torsion class in* mod Λ *such that* $(M, P) \in s\tau\text{-tilt}\,\Lambda$. *Then*

$$\operatorname{Filt}(\operatorname{Fac}(\alpha(\mathcal{T}))) = \mathcal{T}.$$

Proof. We show it for the case where Λ is hereditary. For the general case see [MŠ17, Propostion 3.9]. Since $\mathcal{T}$ is closed under quotients, $\mathrm{Fac}(\alpha(\mathcal{T})) \subset \mathcal{T}$. Let $M = M_\rho \oplus M_\lambda$ be the canonical decomposition of $M = P(\mathcal{T})$. By Remark 3.18, we know that $M_\lambda \in \alpha(\mathcal{T})$. Moreover, given that we have an exact sequence

$$\Lambda \xrightarrow{f} M_0 \to M_1 \to 0$$

where $\mathrm{add}(M_0) = \mathrm{add}(M_\rho)$ and $\mathrm{add}(M_1) = \mathrm{add}(M_\lambda)$, we get that $M_\lambda \in \mathrm{Fac}(\alpha(\mathcal{T}))$. Hence $M \in \mathrm{Fac}(\alpha(\mathcal{T}))$ and $\mathcal{T} = \mathrm{Fac}(M) \subset \mathrm{Fac}(\alpha(\mathcal{T}))$. We conclude that

$$\mathcal{T} = \mathrm{Fac}(\alpha(\mathcal{T})).$$

□

Definition 3.20. Let $\mathcal{W}$ be a wide subcategory of $\mathrm{mod}\,\Lambda$. We say that $\mathcal{W}$ is *left finite* if it is in $\alpha(\text{f-tors}\,\Lambda)$, or equivalently by Proposition 3.16 and Proposition 3.19, if $\mathrm{Filt}(\mathrm{Fac}(\mathcal{W}))$ is a functorially finite torsion class. We denote by $\text{l-wide}\,\Lambda \subset \mathrm{wide}\,\Lambda$ the set of left finite wide subcategories of $\mathrm{mod}\,\Lambda$.

Remark 3.21. The content in Definition 3.20 might be perceived as somewhat unsatisfactory. Indeed, the fact that $\mathcal{W}$ is left finite is not intrinsic to $\mathcal{W}$ as a category, but depends on its embedding in $\mathrm{mod}\,\Lambda$. We will explore how this flaw disappears when we examine the analogous notion in $\mathcal{K}^{[-1,0]}(\mathrm{proj}\,\Lambda)$.

Theorem 3.22. *[IT09, MŠ17] Let Λ be any finite-dimensional algebra. There is a bijection between the sets of functorially finite torsion classes* $\text{f-tors}\,\Lambda$ *and the set of left finite wide subcategories* $\text{l-wide}\,\Lambda$ *given by the maps*

$$\begin{aligned} \text{l-wide}\,\Lambda &\longleftrightarrow \text{f-tors}\,\Lambda \\ \mathcal{W} &\longmapsto \mathrm{Filt}(\mathrm{Fac}(\mathcal{W})) \\ \alpha(\mathcal{T}) &\longleftarrow\!\shortmid \mathcal{T}. \end{aligned}$$

Remark 3.23. Together, Theorem 3.9 and Theorem 3.22 give a correspondence between support τ-tilting pairs and left finite wide subcategories. This correspondence was thoroughly studied by S. Asai in [Asa20], who showed that given any left finite wide subcategory $\mathcal{W}$, its simple modules can be explicitly described from its associated support τ-tilting pair. We recall Asai's results in Chapter 4.

3.1.2 Semistability and τ-tilting theory

In this subsection, we revisit the relationship between the concept of semistability discussed in Chapter 2 and the findings outlined in Section 3.1.1.

Proposition 3.24. *Let $N \in \mathrm{mod}\,\Lambda$ and $X = \begin{array}{c} X^{-1} \\ \downarrow f \\ X^0 \end{array} \in \mathcal{K}^{[-1,0]}(\mathrm{proj}\,\Lambda)$. Then the map*

$$\mathrm{Hom}_\Lambda(X^{-1}, N) \xrightarrow{\mathrm{Hom}_\Lambda(f,N)} \mathrm{Hom}_\Lambda(X^0, N)$$

is an isomorphism if and only if $N \in {}^{\perp}H^{-1}(\nu X) \cap H^0(X)^{\perp}$.

Proof. Let $N \in \operatorname{mod}\Lambda$ and $X = \begin{array}{c} X^{-1} \\ \downarrow f \\ X^{0} \end{array} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Applying $\nu : \operatorname{proj}\Lambda \to \operatorname{inj}\Lambda$, we get exact sequences

$$X^{-1} \xrightarrow{x} X^0 \to H^0(X) \to 0$$

$$0 \to H^{-1}(\nu X) \to \nu X^{-1} \xrightarrow{\nu f} \nu X^0.$$

Now, applying $\operatorname{Hom}_\Lambda(N,-)$, $D\operatorname{Hom}_\Lambda(-,N)$ and using the fact that for any projective module P, we have $\operatorname{Hom}_\Lambda(N,\nu P) \simeq D\operatorname{Hom}_\Lambda(P,N)$, we get the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_\Lambda(N,H^{-1}(\nu X)) \arrow[r] & \operatorname{Hom}_\Lambda(N,\nu X^{-1}) \arrow[r, "\operatorname{Hom}_\Lambda(\nu f,N)"] \arrow[d, "\simeq"] & \operatorname{Hom}_\Lambda(N,\nu X^0) \arrow[d, "\simeq"] & \\
 & D\operatorname{Hom}_\Lambda(X^{-1},N) \arrow[r, "D\operatorname{Hom}_\Lambda(f,N)"] & D\operatorname{Hom}_\Lambda(X^0,N) \arrow[r] & D\operatorname{Hom}_\Lambda(H^0(X),N).
\end{tikzcd}$$

In particular, $\operatorname{Hom}_\Lambda(X^{-1},N) \xrightarrow{\operatorname{Hom}_\Lambda(f,N)} \operatorname{Hom}_\Lambda(X^0,N)$ is an isomorphism if and only if $\operatorname{Hom}_\Lambda(H^0(X),N) = 0 = \operatorname{Hom}_\Lambda(N,H^{-1}(\nu X))$. □

Remark 3.25. For any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, we have that $X \simeq X_M \oplus P[1]$ where X_M is the minimal projective presentation of $M = H^0(X)$. In particular, $H^{-1}(\nu X) = \tau H^0(X) \oplus \nu P$.

Lemma 3.26. *[Yur18, Lemma 3.1] For any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $N \in \operatorname{mod}\Lambda$, we have*

$$\langle [X],[N] \rangle = \dim_{\Bbbk} \operatorname{Hom}_\Lambda(H^0(X),N) - \dim_{\Bbbk} \operatorname{Hom}_\Lambda(N,H^{-1}(\nu X)).$$

Definition 3.27. Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, we say that X is a 2-term *presilting* complex if $\operatorname{Hom}_b(X,X[i]) = 0$ for all $i > 0$. We say that X is *silting* if its presilting and if the smallest thick subcategory of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ containing X is $\mathcal{K}^b(\operatorname{proj}\Lambda)$ itself.

Proposition 3.28. *[AIR14, Proposition 3.3] Let $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a 2-term presilting complex, then X is 2-term silting if and only if $|X| = |\Lambda|$.*

Proposition 3.29. *[Pla13, Lemma 2.6] [DF15, Lemma 3.4] Let $M, N \in \operatorname{mod}\Lambda$ with projective presentations $X_M, X_N \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. If X_M is minimal, then*

$$D\operatorname{Hom}_\Lambda(N,\tau M) \simeq \mathbb{E}_{\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)}(X_M,X_N)$$

where τ is the Auslander-Reiten translation in $\operatorname{mod}\Lambda$.

Proposition 3.30. *[AIR14, Proposition 3.7] Let (M,P) be a support τ-rigid pair, then $X_M \oplus P[1]$ is a presilting complex in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Moreover, if (M,P) is support τ-tilting, then $X_M \oplus P[1]$ is silting.*

Proof. Let (M,P) be a support τ-rigid pair and $X = X_M \oplus P[1]$. By Proposition 3.29 we get that

$$\begin{aligned} \mathbb{E}_{[-1,0]}(X,X) &\simeq \mathbb{E}_{[-1,0]}(X_M,X) \oplus \mathbb{E}_{[-1,0]}(P[1],X) \simeq \\ &\simeq D\operatorname{Hom}_\Lambda(M,\tau M) \oplus \mathbb{E}_{[-1,0]}(P[1],X) = \mathbb{E}_{[-1,0]}(P[1],X). \end{aligned}$$

But $\mathbb{E}_{[-1,0]}(P[1],X) \simeq \operatorname{Hom}_b(P,X) \simeq \operatorname{Hom}_\Lambda(P,M) = 0$, hence X is a presilting object in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Now suppose that (M,P) is support τ-tinting pair, as we recalled in Proposition 1.20, $|X| = |X_M| + |P| = |M| + |P| = |\Lambda|$. By Proposition 3.28, we get the result. □

Proposition 3.24 implies that for any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, the subcategory $\mathscr{W}(X)$ satisfies that

$$\mathscr{W}(X) = {}^{\perp}H^{-1}(\nu X) \cap H^{0}(X)^{\perp}.$$

In particular, to any support τ-rigid pair (M,P) we can associate the category $\mathscr{W}_{(M,P)} = \mathscr{W}(X_M \oplus P[1]) = {}^{\perp}\tau M \oplus \nu P \cap M^{\perp} = {}^{\perp}\tau M \cap P^{\perp} \cap M^{\perp}$. In Chapter 2 we showed that, when $\Bbbk = \mathbb{C}$, $\mathscr{W}_{(M,P)}$ coincides with the wide subcategory $\mathscr{W}_{[X_M \oplus P[1]]}$ of $[X_M \oplus P[1]]$-semistable modules. We thus have two ways of associating a wide subcategory to a support τ-tilting pair (M,P), namely $\mathscr{W}_{(M_\lambda,N)} = {}^{\perp}\tau M_\lambda \cap P^{\perp} \cap M_\lambda^{\perp}$ and $\alpha(\operatorname{Fac}(M))$.

Theorem 3.31. *[Yur18, Theorem 1.3] [BST19, Lemma 4.15]Let Λ be a finite dimensional algebra over and arbitrary field $\Bbbk$. For any support τ-tilting pair (M,P), then*

$$\mathscr{W}_{[X_{M_\lambda} \oplus P[1]]} = \mathscr{W}_{(M_\lambda,P)} = \alpha(\operatorname{Fac}(M)),$$

where M_λ is as in Lemma 3.11. In particular any left finite wide subcategory can be realized as the subcategory of θ-semistable modules for some $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$.

Proof. Let (M,P) be a support τ-tilting pair, then by [AIR14, Corollary 2.13] we know that

$$\operatorname{Fac}(M) = {}^{\perp}\tau M \cap P^{\perp}.$$

Let $M = M_\rho \oplus M_\lambda$ the canonical decomposition of M given by Lemma 3.11 and $\Lambda \xrightarrow{f} M_0 \xrightarrow{g} M_1 \to 0$ the corresponding exact sequence. Since $\operatorname{add}(\tau M_\lambda) \subset \operatorname{add}(\tau M)$, we have that ${}^{\perp}\tau M \subset {}^{\perp}\tau M_\lambda$. For the other inclusion, note that epimorphism g induces a monomorphism $\tau g : \tau M_0 \to \tau M_1$, and thus, for any $N \in {}^{\perp}\tau M_\lambda = {}^{\perp}\tau M_1$, we must have $N \in {}^{\perp}\tau M_\rho = {}^{\perp}\tau M_0$. This tells us that $N \in {}^{\perp}\tau M$. By Proposition 3.17, we conclude that

$$\alpha(\operatorname{Fac}(M)) = \operatorname{Fac}(M) \cap M_\lambda^{\perp} = {}^{\perp}\tau M \cap P^{\perp} \cap M_\lambda^{\perp} = {}^{\perp}\tau M_\lambda \cap P^{\perp} \cap M_\lambda^{\perp},$$

where ${}^{\perp}\tau M_\lambda \cap P^{\perp} \cap M_\lambda^{\perp}$ is presicely $\mathscr{W}_{(M_\lambda,P)}$.

The only thing left to prove its that $\mathscr{W}_{[X_{M_\lambda} \oplus P[1]]} = \mathscr{W}_{(M_\lambda,P)}$. Let $\theta = X_{M_\lambda} \oplus P[1]$ and let $N \in \mathscr{W}_{(M_\lambda,P)}$, by Proposition 3.24 we get that $\langle \theta, [N] \rangle = 0$. Now let $N' \subset N$, by applying the exatct functors $\operatorname{Hom}_\Lambda(X_\lambda^0, -)$ and $\operatorname{Hom}_\Lambda(X_\lambda^{-1} \oplus P, -)$, we get a commutative square

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_\Lambda(X_\lambda^0, N') \arrow[r] \arrow[d, "\operatorname{Hom}_\lambda(x_\lambda, N')"] & \operatorname{Hom}_\Lambda(X_\lambda^0, N) \arrow[d, "\operatorname{Hom}_\Lambda(x_\lambda, N)"] \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(X_\lambda^{-1} \oplus P, N') \arrow[r] & \operatorname{Hom}_\Lambda(X_\lambda^{-1} \oplus P, N)
\end{tikzcd}$$

where the map $\operatorname{Hom}_\Lambda(x_\lambda, N)$ is a isomorphism by Proposition 3.24. In particular, $\operatorname{Hom}_\lambda(x_\lambda, N')$ is injective and thus

$$\dim_\Bbbk \operatorname{Hom}_\Lambda(X_\lambda^0, N') - \dim_\Bbbk \operatorname{Hom}_\Lambda(X_\lambda^{-1} \oplus P, N') = \langle \theta, [N'] \rangle \leq 0,$$

implying that N is θ-semistable. For the other inclusion, let $N \in \operatorname{mod}\Lambda$ be a $[X_{M_\lambda} \oplus P[1]]$-semistable module. Since $\operatorname{Fac}(M_\lambda)$ is a torsion class, there is a short exact sequence

$$0 \to tN \to N \to N/tN \to 0$$

such that $tN \in \operatorname{Fac}(M_\lambda)$ and $N/tN \in M_\lambda^\perp$. By hypothesis $\langle\theta, tN\rangle \leq 0$. As we have seen before, $\operatorname{Fac}(M_\lambda) \subset {}^\perp\tau M_\lambda \cap P^\perp = {}^\perp H^{-1}(\nu X_{M_\lambda} \oplus \nu P[1])$. If $tN \neq 0$, Lemma 3.26 gives that

$$\begin{aligned}\langle\theta, [tN]\rangle &= \dim_{\Bbbk} \operatorname{Hom}_\Lambda(H^0(X_\lambda \oplus P), tN) - \dim_{\Bbbk} \operatorname{Hom}_\Lambda(tN, H^{-1}(\nu X_{M_\lambda} \oplus \nu P[1]))\\ &= \dim_{\Bbbk} \operatorname{Hom}_\Lambda(M_\lambda, tN) > 0\end{aligned}$$

since by definition there exists an epimorphism $M' \to tN$ where $M' \in \operatorname{add}(M_\lambda)$. This implies that $tN = 0$ and $N \in M_\lambda^\perp$. To see that $N \in {}^\perp H^{-1}(\nu X_\lambda \oplus \nu P) = {}^\perp\tau M_\lambda \cap P^\perp$ use a similar argument using that $(\tau M_\lambda \cap P^\perp, \operatorname{Sub}((\tau M_\lambda \oplus P))$ is a torsion class, where for any $L \in \operatorname{mod}\Lambda$, $\operatorname{Sub}(L) = \{N \in \operatorname{mod}\Lambda \mid \exists\, 0 \to N \to L'$ *exact sequence with* $L' \in \operatorname{add}(L)\}$. We conclude that

$$\mathscr{W}_{[X_{M_\lambda} \oplus P[1]]} = \mathscr{W}_{(M_\lambda, P)} = \alpha(\operatorname{Fac}(\alpha)).$$

□

3.2 Mirror structures in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$

Many of the objects introduced in Section 3.1.1 have "mirror" notions in the category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ of projective presentations.

Theorem 3.32. *[AIR14, Theorem 3.2] Let Λ be a finite-dimensional algebra over a field. Then the functor $H^0 : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \to \operatorname{mod}\Lambda$ induces a bijection between*

(1) The set sτ-tilt Λ *of isomorphism classes of basic support τ-tilting pairs in* $\operatorname{mod}\Lambda$.

(i) The set 2-silt Λ *of isomorphism classes of basic silting* 2*-term complexes in* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Theorem (3.42). *[PZ23, Theorem 3.6] There is a well defined map*

$$\Phi : \operatorname{cotor}\Lambda \to \operatorname{tors}\Lambda$$

between the set cotor Λ *of cotorsion pairs in* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *and the set* tors Λ *of torsion pairs in* $\operatorname{mod}\Lambda$. *This map restricts to a bijection between*

(2) Functorially finite torsion pairs in $\operatorname{mod}\Lambda$.

(ii) Complete cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

The goal of this section is to show that Theorem 3.1 has an analog in the category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ of projective presentations which are compatible with the two previous correspondences. In Section 3.2.1, we revisit the concept of 2-term (pre)silting complex and the notion of a cotorsion pair for a general extriangulated category. We present the results of T. Adachi and M. Tsukamoto, who established an explicit bijection Ψ between cotorsion pairs in a general extriangulated category $\mathcal{K}$ and its silting subcategories. When $\mathcal{K} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, this yields

Theorem (3.43). *[AT22, Theorem 5.7] The following sets are in one-to-one correspondence:*

(i) Isomorphism classes of basic 2-term silting objects in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

(ii) Complete cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

This correspondence takes a complete cotorsion pair $(\mathcal{X},\mathcal{Y})$ *and sends it to* $\Psi((\mathcal{X},\mathcal{Y})) = U$, *where* U *is a basic additive generator of the silting category* $\mathcal{X}\cap\mathcal{Y}$.

Then, in Section 3.2.2, we introduce the set $\operatorname{thick}\Lambda$ of *thick subcategories* (see Section 1.2) of the extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and show that those who have enough injectives are in bijection with the set of complete cotorsion pairs.

Theorem (3.51). *Let* Λ *be a finite-dimensional* $\Bbbk$*-algebra. There exist well defined maps*

$$\begin{tikzcd}\operatorname{cotor}\Lambda \arrow[r, shift left, "\beta"] & \operatorname{thick}\Lambda \arrow[l, shift left, "I"]\end{tikzcd}$$

between the set of cotorsion pairs $\operatorname{cotor}\Lambda$ *and that of thick subcategories* $\operatorname{thick}\Lambda$ *of* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, *such that when restricted to the set* $\text{c-cotor}\,\Lambda$ *of complete cotorsion pairs and the set* $\text{inj-thick}\,\Lambda$ *of thick subcategories with enough injectives, they are inverse of each other.*

Finally, in Section 3.2.3, we show that thick subcategories are a natural analog of wide subcategories. We use as inspiration the description of left wide subcategories of $\operatorname{mod}\Lambda$ as subcategories of semistable modules as seen in Section 3.1.2. The main result of this subsection is the following .

Theorem (3.72). *Let* Λ *be a finite-dimensional* $\Bbbk$*-algebra. There exist inclusion-reversing maps*

$$\begin{tikzcd}\operatorname{wide}\Lambda \arrow[r, shift left, "\mathscr{T}"] & \operatorname{thick}\Lambda \arrow[l, shift left, "\mathscr{W}"]\end{tikzcd}$$

such that, when restricted to thick subcategories with enough injectives and the set $\text{l-wide}\,\Lambda$ *of left finite wide subcategories, they make the following diagram commute*

$$\begin{tikzcd}
 & & \text{2-silt}\,\Lambda \arrow[dr, "\operatorname{thick}(U_\rho)"] & & \\
 & \text{c-cotor}\,\Lambda \arrow[ur, "\Psi"] \arrow[rr, "\beta"] \arrow[ddd, "\Phi"] & & \text{inj-thick}\,\Lambda \arrow[ddd, "\mathscr{W}"] & \\
\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) & & & & \\
 & & & & \operatorname{mod}\Lambda \\
 & \text{f-tors}\,\Lambda \arrow[rr, "\alpha"] & & \text{l-wide}\,\Lambda &
\end{tikzcd}$$

In particular, $\mathscr{W}$ *and the map taking any* $U\in\text{2-silt}\,\Lambda$ *to the thick category* $\operatorname{thick}(U_\rho)\in\text{inj-thick}\,\Lambda$ *are bijective. Here,* U_ρ *is the basic direct summand of* U *satisfying* $\operatorname{add}(U_\rho)=\operatorname{add}(U')$, *where* U' *is such that* $U'\to\Lambda[1]$ *is a minimal right* $\operatorname{add}(U)$*-approximation of* $\Lambda[1]$.

Putting all previous theorems together we get the main result of this chapter.

Corollary 3.33. *There are explicit bijections between*

(i) Isomorphism classes of basic 2*-term silting objects in* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$*.*

(ii) Complete cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$*.*

(iii) Thick subcategories in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *with enough injectives.*

These bijections are compatible with those in Theorems 3.1, 3.32, 3.42, and 3.43. In other words, the following diagram commutes

$$\begin{tikzcd}
& & \text{2-silt}\,\Lambda \arrow[rd, "\mathrm{thick}(U_\rho)"] \arrow[ddddd, "H^0"] & & \\
& \text{c-cotor}\,\Lambda \arrow[ru, "\Psi"] \arrow[rr, "\beta"] \arrow[ddddd, "\Phi"'] & & \text{inj-thick}\,\Lambda \arrow[ddddd, "\mathscr{W}"] & \\
\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) & & & & \\
{} \arrow[r, dash, dashed] & {} \arrow[rr, dash, dashed] & & {} \arrow[r, dash, dashed] & {} \\
& & & & \operatorname{mod}\Lambda \\
& & \text{s}\tau\text{-tilt}\,\Lambda \arrow[ld, "\mathrm{Fac}"'] \arrow[rd] & & \\
& \text{f-tors}\,\Lambda \arrow[rr, "\alpha"] & & \text{l-wide}\,\Lambda &
\end{tikzcd}$$

3.2.1 Cotorsion pairs and τ-tilting theory

In this section we recall some results and tools used in the study of cotorsion pairs and silting subcategories in a general extriangulated category $\mathcal{K}$. Most of these results first appeared in the context of extriangulated categories in [AT22]. We will often apply these results to the category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. From now on, we suppose that all our subcategories are full and stable under isomorphisms.

Definition 3.34. Let $\mathcal{K}$ be an extriangulated category and $\mathcal{X}$ and $\mathcal{Y}$ two subcategories of $\mathcal{K}$. We denote by $\operatorname{Cone}(\mathcal{X},\mathcal{Y})$ the full subcategory whose objects are those Z such that there exists a conflation $X \rightarrowtail Y \twoheadrightarrow Z$, where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Dually, we say that $Z \in \operatorname{Cocone}(\mathcal{X},\mathcal{Y})$ if there exists a conflation $Z \rightarrowtail X \twoheadrightarrow Y$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We define as well:

1. $\mathcal{X}^{\wedge}_{-1} = 0$, $\mathcal{X}^{\wedge}_{m} = \operatorname{Cone}(\mathcal{X}^{\wedge}_{m-1}, \mathcal{X})$ $\forall m \in \mathbb{Z}_{\geq 0}$, and $\mathcal{X}^{\wedge} = \bigcup_{m\in\mathbb{Z}_{\geq 0}} \mathcal{X}^{\wedge}_{m}$.

2. $\mathcal{X}^{\vee}_{-1} = 0$, $\mathcal{X}^{\vee}_{m} = \operatorname{Cocone}(\mathcal{X}, \mathcal{X}^{\vee}_{m-1})$ $\forall m \in \mathbb{Z}_{\geq 0}$, and $\mathcal{X}^{\vee} = \bigcup_{m\in\mathbb{Z}_{\geq 0}} \mathcal{X}^{\vee}_{m}$.

Let $\mathcal{H}$ be a subcategory of an extriangulated category $\mathcal{K}$, recall that $\operatorname{thick}(\mathcal{H})$ is the smallest thick subcategory containing $\mathcal{H}$. That is, $\operatorname{thick}(\mathcal{H})$ is the smallest subcategory which contains $\mathcal{H}$, is full, additive and closed under extensions, cones and cocones.

Definition 3.35. Let $\mathcal{K}$ be an extriangulated category. We say that a subcategory $\mathcal{U} \subset \mathcal{K}$ is *presilting* if it is closed under direct sums and summands and $\mathbb{E}^i(\mathcal{U},\mathcal{U}) = 0$ for all $i > 0$. We say that $\mathcal{U}$ is *silting* if $\operatorname{thick}(\mathcal{U}) = \mathcal{K}$. An object $U \in \mathcal{K}$ is (pre)silting if the category $\operatorname{add}(U)$ satisfies this property. We denote by $\operatorname{silt}\mathcal{K}$ the set of isomorphism classes of basic silting objects in $\mathcal{K}$.

Remark 3.36. Recall that in a triangulated category $\mathcal{D}$ with shift functor Σ, an object $U \in \mathcal{D}$ is presilting if $\operatorname{Hom}_{\mathcal{D}}(U, \Sigma^i U) = 0$ for all $i > 0$. It is silting if $\operatorname{thick}_{\mathcal{D}}(U) = \mathcal{D}$, where $\operatorname{thick}_{\mathcal{D}}(U)$ denotes the smallest triangulated subcategory of $\mathcal{D}$ containing U. For a 2-term complex $U \in \mathcal{K}^b(\operatorname{proj}\Lambda)$, being presilting in both $\mathcal{K}^b(\operatorname{proj}\Lambda)$ and the extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ are equivalent since $\operatorname{Hom}_b(U, U[i]) = 0$ for all $i \geq 2$. If U is silting in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, then $\Lambda \in \operatorname{thick}_{[-1,0]}(U) \subset \operatorname{thick}_b(U)$ and thus U is silting in the triangulated category $\mathcal{K}^b(\operatorname{proj}\Lambda)$. Suppose now that $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is silting as an object in the triangulated category in $\mathcal{K}^b(\operatorname{proj}\Lambda)$. There exists a triangle

$$\Lambda \to U' \to U'' \xrightarrow{f} \Lambda[1]$$

where f is a minimal right $\operatorname{add}(U)$-approximation of Λ and such that $\operatorname{add}(U' \oplus U'') = \operatorname{add}(U)$ [AI12, Proposition 2.24]. Thus $\Lambda \in \operatorname{Cocone}(U, U) \subset \operatorname{thick}_{[-1,0]}(X)$ and U is silting in the extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Consequently, U is a 2-term silting object of $\mathcal{K}^b(\operatorname{proj}\Lambda)$ if and only if it is a silting object in the extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Hence both terminologies can be used interchangeably, which we do in this text. Moreover, we will write $2\text{-silt}\,\Lambda = \operatorname{silt}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Proposition 3.37. *[AT22, Lemma 5.3] Let $\mathcal{K}$ be an extriangulated category and let $\mathcal{V} \subset \mathcal{K}$ be a silting subcategory. If $\mathcal{U}$ is a presilting subcategory with $\mathcal{V} \subset \mathcal{U}$, then $\mathcal{U} = \mathcal{V}$.*

Proposition 3.38. *[AT22, Proposition 5.4] Let $\mathcal{K}$ be an extriangulated category that contains a silting object. Then each silting category admits an additive generator. Moreover, if $\mathcal{K}$ is a Krull-Schmidt category, then $U \mapsto \operatorname{add} U$ gives a bijection between $\operatorname{silt}\mathcal{K}$ and the set of silting subcategories.*

Proposition 3.39. *[AT22, Proposition 4.10] Let $\mathcal{U}$ be a presilting subcategory of $\mathcal{K}$. Then the following statements hold.*

1. *$\mathcal{U}^{\vee}$ is the smallest subcategory containing $\mathcal{U}$ and closed under extensions, cocones and direct summands. Moreover, if $\mathcal{U}$ is closed under cones, then $\mathcal{U}^{\vee} = \operatorname{thick}(\mathcal{U})$.*
2. *$\mathcal{U}^{\wedge}$ is the smallest subcategory containing $\mathcal{U}$ and closed under extensions, cones and direct summands. Moreover, if $\mathcal{U}$ is closed under cocones, then $\mathcal{U}^{\wedge} = \operatorname{thick}(\mathcal{U})$.*

Definition 3.40. [PZ23, Definition 1.7] Let $\mathcal{K}$ be an extriangulated category. We say that a pair of subcategories $(\mathcal{X}, \mathcal{Y})$ is a *cotorsion pair* if they are both full and additive and they satisfy

1. $\mathbb{E}(X, \mathcal{Y}) = 0$ if and only if $X \in \mathcal{X}$.
2. $\mathbb{E}(\mathcal{X}, Y) = 0$ if and only if $Y \in \mathcal{Y}$.

In other words $\mathcal{Y} = \mathcal{X}^{\perp_1} = \{Y \in \mathcal{K} \mid \mathbb{E}(X,Y) = 0 \ \forall\ X \in \mathcal{X}\}$ and $\mathcal{X} = {}^{\perp_1}\mathcal{Y} = \{X \in \mathcal{K} \mid \mathbb{E}(X,Y) = 0 \ \forall\ Y \in \mathcal{Y}\}$. We denote by $\operatorname{cotor}\mathcal{K}$ the set of all cotorsion pairs in $\mathcal{K}$. We say that $(\mathcal{X},\mathcal{Y})$ is *complete* [NP19, Definition 4.1], if additionally $\mathcal{K} = \operatorname{Cone}(\mathcal{Y},\mathcal{X}) = \operatorname{Cocone}(\mathcal{Y},\mathcal{X})$. We denote by $\text{c-cotor}\,\mathcal{K} \subset \operatorname{cotor}\mathcal{K}$ the subset of complete cotorsion pairs of $\mathcal{K}$.

Remark 3.41. As we have noted before, when $\mathcal{K} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ it is always true that $\mathbb{E}^2(X,Y) = 0$ for all $X, Y \in \mathcal{K}$. In particular, $\mathbb{E}^2(\mathcal{X},\mathcal{Y}) = 0$ for all cotorsion pairs $(\mathcal{X},\mathcal{Y})$. We say that a cotorsion pair is *hereditary* [LZ20, Definition 4.1] when it satisfies this property. We remark as well that all projective objects $\operatorname{add}(\Lambda)$ must lie in $\mathcal{X}$, all injective objects $\operatorname{add}(\Lambda[1])$ belong to $\mathcal{Y}$ and since $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \operatorname{Cone}(\operatorname{add}(\Lambda),\operatorname{add}(\Lambda)) = \operatorname{Cocone}(\operatorname{add}(\Lambda[1]),\operatorname{add}(\Lambda[1]))$ we have that $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \mathcal{X}^{\wedge} = \mathcal{Y}^{\vee}$. In a general extriangulated category $\mathcal{K}$, we say that $(\mathcal{X},\mathcal{Y})$ is *bounded* [AT22] if it satisfies that $\mathcal{K} = \mathcal{X}^{\wedge} = \mathcal{Y}^{\vee}$.

Theorem 3.42. *[PZ23, Theorem 3.6] Let Λ be a finite-dimensional $\Bbbk$-algebra. There are well defined maps:*

$$\begin{tikzcd}
\operatorname{cotor}\Lambda \arrow[r, "\Phi"] & \operatorname{tors}\Lambda \\
\text{c-cotor}\,\Lambda \arrow[u, phantom, "\cup|"] & \text{f-tors}\,\Lambda \arrow[l, "\Theta"'] \arrow[u, phantom, "\cup|"]
\end{tikzcd}$$

given by

$$\Phi((\mathcal{X},\mathcal{Y})) = (H^0(\mathcal{Y}), H^0(\mathcal{Y})^{\perp})$$

and

$$\Theta(\mathcal{T},\mathcal{F}) = ({}^{\perp_1}\mathcal{Z},\mathcal{Z})$$

where $\mathcal{Z} = \left(H^0\right)^{-1}(\mathcal{T})$. Moreover, Θ and Φ are inverse to each other when restricted to c-cotor Λ *and* f-tors Λ*.*

The following is an application of Theorem 5.7 in [AT22] to the extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. The statement follows from the fact that, in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, all complete cotorsion pairs are hereditary and bounded. The original theorem gives a bijection between c-cotor Λ and the set of its silting subcategories. Since $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is Krull-Schmidt, Proposition 3.38 allows us to state the bijection in terms of 2-silt Λ.

Theorem 3.43. *[AT22, Theorem 5.7] There is a one-to-one correspondence*

$$\begin{tikzcd}
\text{c-cotor}\,\Lambda \arrow[r, shift left, "\Psi"] & \text{2-silt}\,\Lambda \arrow[l, shift left, "\Xi"]
\end{tikzcd}$$

given by the maps

$$\Xi(U) = (\operatorname{add}(U)^{\vee}, \operatorname{add}(U)^{\wedge})$$

and

$$\Psi((\mathcal{X},\mathcal{Y})) = U$$

where U is a basic additive generator of the silting category $\mathcal{X} \cap \mathcal{Y}$.

Definition 3.44. Let $\mathcal{K}$ be an extriangulated category and let $\mathcal{X} \subset \mathcal{K}$. We say that $\mathcal{X}$ is a *resolving subcategory* of $\mathcal{K}$ if $\mathcal{K} = \operatorname{Cone}(\mathcal{K}, \mathcal{X})$ and it is closed under extensions, cocones and direct summands. We write $\text{f-res}\,\mathcal{K} \subset \operatorname{res}\mathcal{K}$ for the sets of contravariantly finite resolving subcategories and resolving subcategories of $\mathcal{K}$, respectively.

Remark 3.45. When $\mathcal{K}$ has enough projectives, we can swap the condition of $\mathcal{X}$ satisfying $\mathcal{K} = \operatorname{Cone}(\mathcal{K}, \mathcal{X})$ in the previous definition by $\mathcal{X}$ containing all projective objects. Indeed, if $\operatorname{proj}\mathcal{K} \subset \mathcal{X}$, since all objects C admit a deflation $P_C \twoheadrightarrow C$ with $P_C \in \operatorname{proj}\mathcal{K}$, then $\mathcal{K} \subset \operatorname{Cone}(\mathcal{K}, \operatorname{proj}\mathcal{K}) \subset \operatorname{Cone}(\mathcal{K}, \mathcal{X})$. Reciprocally, if $\mathcal{K} = \operatorname{Cone}(\mathcal{K}, \mathcal{X})$, then for all $P \in \operatorname{proj}\mathcal{K}$ there is $X_P \in \mathcal{X}$ and a conflation $Y \rightarrowtail X_P \twoheadrightarrow P$. But since P is projective, this conflation must split and $P \in \mathcal{X}$, as $\mathcal{X}$ is closed under direct summands.

Suppose now that $\mathcal{K} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and let $\mathcal{X}$ be a resolving and contravariantly finite subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then for every object $C \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ a right $\mathcal{X}$-approximation of C must be a deflation. Indeed, Let $X \xrightarrow{c} C$ be an $\mathcal{X}$-approximation of C. Since $\mathcal{X}$ is resolving, there exists a conflation $Y \overset{f}{\rightarrowtail} X' \overset{g}{\twoheadrightarrow} C$ such that $Y \in \mathcal{K}$ and $X' \in \mathcal{X}$. Thus, there exists a map $X' \xrightarrow{x} X$ such that $g = c \cdot x$. By the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, we have a triangle $\operatorname{Cone}(x) \to Y[1] \to \operatorname{Cone}(c)$ such that the following diagram commutes

$$\begin{tikzcd}
X' \arrow[r, "x"] \arrow[d, equal] & X \arrow[r] \arrow[d, "c"] & \operatorname{Cone}(x) \arrow[d] \\
X' \arrow[r, "g"] & C \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
 & \operatorname{Cone}(c) \arrow[r, equal] & \operatorname{Cone}(c)
\end{tikzcd}$$

But since $\operatorname{Cone}(x) \in \mathcal{K}^{[-2,0]}(\operatorname{proj}\Lambda)$ and $Y[1] \in \mathcal{K}^{[-2,-1]}(\operatorname{proj}\Lambda)$, $\operatorname{Cone}(c)$ must be in $\mathcal{K}^{[-2,0]}(\operatorname{proj}\Lambda) \cap \mathcal{K}^{[-3,-1]}(\operatorname{proj}\Lambda) = \mathcal{K}^{[-2,-1]}(\operatorname{proj}\Lambda)$. Then the triangle $\operatorname{Cone}(c)[-1] \to X \xrightarrow{c} C$ lies in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and c is a deflation. Since $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfies WIC, is Krull-Schmidt and Hom-finite, we can also find an approximation that is minimal.

Proposition 3.46. *[AT22, Proposition 5.15] Let $\mathcal{K}$ be a Krull-Schmidt, Hom-finite extriangulated category satisfying WIC (Definition 1.32) and having enough projectives and injectives. If $(\mathcal{X}, \mathcal{Y})$ is a hereditary complete cotorsion pair, then $\mathcal{X}$ is a contravariantly finite resolving subcategory of $\mathcal{K}$. Reciprocally, if $\mathcal{X} \in \text{f-res}\,\mathcal{K}$, then $(\mathcal{X}, \mathcal{X}^{\perp_1})$ is a complete cotorsion pair.*

Recall that there is a one-to-one correspondence between the set $\text{2-silt}\,\Lambda$ of isomorphism classes of 2-term silting objects in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and the set $\text{s}\tau\text{-tilt}\,\Lambda$ of support τ-tilting basic modules in $\operatorname{mod}\Lambda$ given by the map $H^0 : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \to \operatorname{mod}\Lambda$ (Theorem 3.32). Moreover, the map $M \mapsto \vartheta(M) = (\operatorname{Fac}(M), M^{\perp})$ gives a correspondence between the sets $\text{s}\tau\text{-tilt}\operatorname{mod}\Lambda$ and $\text{f-tors}\,\Lambda$ (Theorem 3.1). The following result shows that these bijections are compatible to the ones described in Theorem 3.42 and Theorem 3.43.

Proposition 3.47. *Let Λ be a finite-dimensional $\mathbb{k}$-algebra and consider the bijections $\Phi : \text{c-cotor}\,\Lambda \to \text{f-tors}\,\Lambda$ of Theorem 3.42, as well as $\Psi : \text{c-cotor}\,\Lambda \to \text{2-silt}\,\Lambda$ of Theorem 3.43. The following diagram*

$$\begin{tikzcd}
 & \text{c-cotor}\,\Lambda \arrow[r, "\Psi"] \arrow[dddd, "\Phi"] & \text{silt}\,\Lambda \arrow[dddd, "H^0"] & \\
\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) & & & \\
{} \arrow[rrr, dash, dashed] & & & {} \\
 & & & \operatorname{mod}\Lambda \\
 & \text{f-tors}\,\Lambda & s\tau\text{-tilt}\,\Lambda \arrow[l, "\vartheta"] &
\end{tikzcd}$$

commutes.

Proof. Let $(\mathcal{X}, \mathcal{Y})$ be a complete cotorsion pair in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. By Proposition 3.46, since $\mathcal{X}$ is contravariantly finite and resolving, the complex $\Lambda[1] = \begin{smallmatrix}\Lambda\\ \downarrow\\ 0\end{smallmatrix}$ admits a conflation

$$U_{\mathcal{Y}} \overset{u}{\rightarrowtail} U_{\mathcal{X}} \overset{\pi_{\mathcal{X}}}{\twoheadrightarrow} \Lambda[1] \tag{3.2.1}$$

where the corresponding deflation is a minimal right $\mathcal{X}$-approximation and $U_{\mathcal{Y}} \in \mathcal{Y}$ by Lemma 1.34. Since $\mathcal{Y}$ is closed under extensions and $\Lambda[1] \in \operatorname{inj}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \subset \mathcal{Y}$, we get that $U_{\mathcal{X}} \in \mathcal{X} \cap \mathcal{Y}$. Moreover, since the sequence $\Lambda \overset{i_{\mathcal{Y}}}{\rightarrowtail} U_{\mathcal{Y}} \overset{u}{\twoheadrightarrow} U_{\mathcal{X}}$ is also a conflation, $\mathcal{X}$ is closed under extensions and $\Lambda \in \operatorname{proj}\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \subset \mathcal{X}$, then $U_{\mathcal{Y}} \in \mathcal{X} \cap \mathcal{Y}$. This implies that $\operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}}) \subset \mathcal{X} \cap \mathcal{Y}$, and since $\Lambda \in \operatorname{thick}(\operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}}))$, we obtain that

$$\operatorname{thick}(\operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})) = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda).$$

By Proposition 3.37, we have that $\mathcal{X} \cap \mathcal{Y} = \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})$, which in turn gives that $\Psi((\mathcal{X}, \mathcal{Y})) = U_{\mathcal{X}\cap\mathcal{Y}}$, where $U_{\mathcal{X}\cap\mathcal{Y}}$ is the basic object such that $\operatorname{add}(U_{\mathcal{X}\cap\mathcal{Y}}) = \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})$.

Let $\mathcal{T} = H^0(\mathcal{Y})$ be the torsion class associated to $\Phi((\mathcal{X}, \mathcal{Y}))$. Applying H^0 to the conflation (3.2.1), we get the exact sequence $H^0(U_{\mathcal{Y}}) \to H^0(U_{\mathcal{X}}) \to 0$. Since $\mathcal{T}$ is closed under quotients, then $\operatorname{Fac}(H^0(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})) \subset \mathcal{T}$. On the other hand, by Theorem 3.43, we know that $\mathcal{Y} = \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})^{\wedge}$, in particular, $\forall\ Y \in \mathcal{Y}$ there exists a conflation $Y' \rightarrowtail U \twoheadrightarrow Y$ where $Y' \in \mathcal{Y}$ and $U \in \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})$. Applying again the functor H^0, we get the exact sequence

$$H^0(Y') \to H^0(U) \to H^0(Y) \to 0$$

which implies that $H^0(Y) \in \operatorname{Fac} H^0(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})$ for all $Y \in \mathcal{Y}$. Then $\mathcal{T} = \operatorname{Fac} H^0(U_{\mathcal{X}} \oplus U_{\mathcal{Y}}) = \operatorname{Fac} H^0(U_{\mathcal{X}\cap\mathcal{Y}})$, which gives the result. □

Corollary 3.48. *For any complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $K^{[-1,0]}(\operatorname{proj}\Lambda)$, there exist conflations*

$$U_{\mathcal{Y}} \overset{u}{\rightarrowtail} U_{\mathcal{X}} \overset{\pi_{\mathcal{X}}}{\twoheadrightarrow} \Lambda[1]$$

$$\Lambda \xrightarrow{\iota_{\mathcal{Y}}} U_{\mathcal{Y}} \overset{u}{\rightarrowtail} U_{\mathcal{X}}$$

where

- *(i) $U_{\mathcal{X}} \in \mathcal{X}$ and $U_{\mathcal{Y}} \in \mathcal{Y}$;*
- *(ii) $U_{\mathcal{X}} \oplus U_{\mathcal{Y}}$ is a silting object such that $\mathcal{X} \cap \mathcal{Y} = \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})$;*

(iii) $\pi_{\mathcal{X}}$ *is a minimal right* $\mathcal{X}$*-approximation of* $\Lambda[1]$.

Remark 3.49. When $\mathcal{X} = (\operatorname{add} U)^{\vee}$ with $U \in 2\text{-silt}\,\Lambda$, then $U_1 \twoheadrightarrow \Lambda[1]$ is a minimal right U-approximation if and only if it is a minimal right $\mathcal{X}$-approximation. Indeed, by the proof of Proposition 3.47, we know that $\operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}}) = \mathcal{X} \cap \mathcal{Y} = \operatorname{add} U$, which implies that $\pi_{\mathcal{X}} : U_{\mathcal{X}} \twoheadrightarrow \Lambda[1]$ is a minimal right U-approximation since $\operatorname{add} U \subset \mathcal{X}$. Consider now $\pi : U_1 \twoheadrightarrow \Lambda[1]$ a minimal right U-approximation. Since $U_1 \in \operatorname{add} U \subset \mathcal{X}$, there exists a map $f : U_1 \to U_{\mathcal{X}}$ such that $\pi = \pi_{\mathcal{X}} \circ f$. But $U_{\mathcal{X}} \in \operatorname{add} U$, so there is $g : U_{\mathcal{X}} \to U_1$ such that $\pi_{\mathcal{X}} = \pi \circ g$. Since $\pi = \pi \circ (g \circ f)$ and π is minimal, $g \circ f$ must be an isomorphism. Using that $\pi_{\mathcal{X}}$ is minimal as well, $f \circ g$ is also an isomorphism such that $\pi_{\mathcal{X}} = \pi_{\mathcal{X}} \circ (f \circ g)$. We conclude that $U_{\mathcal{X}}$ and U_1 are isomorphic.

Remark 3.50. Note that Corollary 3.48 is the analog in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ of Lemma 3.11. Furthermore, for $M = P(H^0(\mathcal{Y}))$, we have that $\operatorname{add}(M_\rho) = \operatorname{add}(H^0(U_{\mathcal{Y}}))$ and $\operatorname{add}(M_\lambda) = \operatorname{add}(H^0(U_{\mathcal{X}}))$. Moreover, if X_{M_ρ} and X_{M_λ} are minimal projective presentations of M_ρ and M_λ respectively, then $\operatorname{add}(X_{M_\rho}) = \operatorname{add}(U_{\mathcal{Y}})$ and $\operatorname{add}(U_{\mathcal{X}}) = \operatorname{add}(X_{M_\lambda} \oplus P[1])$, where $P \in \operatorname{proj}\Lambda$ is the unique projective module satisfying that (M, P) is a support τ-tilting pair.

3.2.2 Thick subcategories and cotorsion pairs

The goal of this section is to prove the following result:

Theorem 3.51. *Let* Λ *be a finite-dimensional* $\Bbbk$*-algebra. Let* $\operatorname{thick}\Lambda$ *be the set of thick subcategories of* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *and* $\operatorname{cotor}\Lambda$ *be the set of cotorsion pairs in* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$*. There exist maps*

$$\begin{tikzcd} \operatorname{cotor}\Lambda \arrow[r, shift left, "\beta"] & \operatorname{thick}\Lambda \arrow[l, shift left, "I"] \end{tikzcd}$$

such that when restricted to complete cotorsion pairs and thick subcategories with enough injectives, they are inverse of each other.

Proposition 3.52. *Let* Λ *be a finite-dimensional* $\Bbbk$*-algebra. There exists a well defined map*

$$\operatorname{res}\Lambda \xrightarrow{\beta} \operatorname{thick}\Lambda$$

which takes any $\mathcal{X} \in \operatorname{res}\Lambda$ *and sends it to*

$$\beta(\mathcal{X}) = \{X \in \mathcal{X} \mid \forall \text{ conflation } X \rightarrowtail X' \twoheadrightarrow X'' \text{ such that } X' \in \mathcal{X}, \text{ then } X'' \in \mathcal{X}\}.$$

Proof. Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. First, we prove that $\beta(\mathcal{X})$ is closed under direct summands. Suppose $X = X' \oplus X'' \in \beta(\mathcal{X}) \subset \mathcal{X}$, then X' and X'' are in $\mathcal{X}$ since $\mathcal{X}$ is closed under direct summands. Let $X' \overset{a}{\rightarrowtail} A \twoheadrightarrow B$ be a conflation with $A \in \mathcal{X}$, then $X' \oplus X'' \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 0 & 1_{X''} \end{smallmatrix}\right)} A \oplus X'' \twoheadrightarrow B$ is also a conflation with $A \oplus X'' \in \mathcal{X}$, which implies that $B \in \mathcal{X}$ since $X \in \beta(\mathcal{X})$. Thus, $X' \in \beta(\mathcal{X})$.

Next, we prove that $\beta(\mathcal{X})$ is closed under extensions. Consider a conflation $X \rightarrowtail X' \twoheadrightarrow X''$ in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $X, X'' \in \beta(\mathcal{X})$. Since $\mathcal{X}$ is closed under extensions, X' is in $\mathcal{X}$. Take $X' \rightarrowtail A \twoheadrightarrow B$ with $A \in \mathcal{X}$. By the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, there exists $C \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[d, equal] & X' \arrow[r, two heads] \arrow[d, tail] & X'' \arrow[d, tail] \\
X \arrow[r, tail] & A \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\
 & B \arrow[r, equal] & B
\end{tikzcd}$$

commutes and such that the last column and the middle row are conflations. Since $X \in \beta(\mathcal{X})$, C must be in $\mathcal{X}$. But $X'' \in \beta(\mathcal{X})$ as well, so $B \in \mathcal{X}$. This implies that $X' \in \beta(\mathcal{X})$ and $\beta(\mathcal{X})$ is closed under extensions.

We now prove that $\beta(\mathcal{X})$ is closed under cones. Let $X \rightarrowtail X' \twoheadrightarrow X''$ be a conflation with $X, X' \in \beta(\mathcal{X})$. In particular, $X' \in \mathcal{X}$, so $X'' \in \mathcal{X}$ by definition of $\beta(\mathcal{X})$. Consider a conflation $X'' \rightarrowtail A \twoheadrightarrow B$ with $A \in \mathcal{X}$. Since $\operatorname{Hom}_b(B[-1], X[1]) = 0$, there exists $h : B[-1] \to X'$ and $D \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, such that

$$\begin{tikzcd}
 & B[-1] \arrow[r, equal] \arrow[d, "h"] & B[-1] \arrow[d] \arrow[rd, "0"] & \\
X \arrow[r, tail] \arrow[d, equal] & X' \arrow[r, two heads] \arrow[d, tail] & X'' \arrow[r] \arrow[d, tail] & X[1] \\
X \arrow[r, tail] & D \arrow[r, two heads] \arrow[d, two heads] & A \arrow[d, two heads] & \\
 & B \arrow[r, equal] & B &
\end{tikzcd} \tag{3.2.2}$$

is a commutative diagram where the second row and column are conflations. Since $\mathcal{X}$ is closed under extensions and $X, A \in \mathcal{X}$, we have that $D \in \mathcal{X}$. Likewise, B must be in $\mathcal{X}$, since $X' \in \beta(\mathcal{X})$, proving that $\beta(\mathcal{X})$ is closed under cones.

In order to prove that $\beta(\mathcal{X})$ is closed under cocones, take now $X \rightarrowtail X' \twoheadrightarrow X''$ a conflation such that $X', X'' \in \beta(\mathcal{X})$. Since $\mathcal{X}$ is resolving, it is closed under cocones and thus, $X \in \mathcal{X}$. Take $X \rightarrowtail A \twoheadrightarrow B$ a conflation in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with $A \in \mathcal{X}$. Using the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, we get the commutative diagram

$$\begin{tikzcd}
X''[-1] \arrow[r] \arrow[d, equal] & X \arrow[r, tail] \arrow[d, tail] & X' \arrow[r, two heads] \arrow[d, tail] & X'' \arrow[d, equal] \\
X''[-1] \arrow[r] & A \arrow[r, tail] \arrow[d, two heads] & C \arrow[r, two heads] \arrow[d, two heads] & X'' \\
 & B \arrow[r, equal] & B &
\end{tikzcd}$$

Since both $\mathcal{K}$ and $\mathcal{X}$ are closed under extensions, $C \in \mathcal{X}$. Using that $X' \in \beta(\mathcal{X})$, we get that $B \in \mathcal{X}$, which gives that $X \in \beta(\mathcal{X})$, so it is closed under cocones. $\square$

Remark 3.53. Proposition 3.52 holds if we substitute $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ by any hereditary extriangulated category $\mathcal{K}$. Indeed, instead of referring to the octahedral axiom, we could have referred to the axioms ET4 and its dual in the definition of extriangulated category (Definition 1.8). The existence of an object D and a diagram as in 3.2.2 is guaranteed by Proposition 1.29 *iv*).

Proposition 3.54. *Let $\mathcal{C} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be an extension-closed subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ that contains the zero object. Then*

$$\iota(\mathcal{C}) = \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid \exists \text{ an inflation } X \rightarrowtail C \text{ with } C \in \mathcal{C}\}$$

is a resolving subcategory. Moreover, if $\mathcal{C}' \subset \mathcal{C}$ is also closed under extensions, then $\iota(\mathcal{C}') \subset \iota(\mathcal{C})$.

Proof. Let $\mathcal{C}$ be a extension-closed subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ containing the zero object. Note that for $P \in \operatorname{proj}\Lambda$, $P \to 0 \to P[1]$ is always a conflation. Since $0 \in \mathcal{C}$, we have that $\operatorname{proj}\Lambda \subset \iota(\mathcal{C})$. That $\iota(\mathcal{C})$ is closed under cocones and direct summands follows directly from the definition of ι. Take $X \rightarrowtail X' \twoheadrightarrow X''$ a conflation where $X, X'' \in \iota(\mathcal{C})$. In particular, there is a conflation $X'' \rightarrowtail C \twoheadrightarrow W$ where $C \in \mathcal{C}$. Using that $\operatorname{Hom}_b(W[-1], X[1]) = 0$ and the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, we get the commutative diagram

$$\begin{tikzcd}
 & W[-1] \arrow[r, equal] \arrow[d] & W[-1] \arrow[d] \arrow[dr, dashed, "0"] & \\
X \arrow[r, tail] \arrow[d, equal] & X' \arrow[r, two heads] \arrow[d, tail] & X'' \arrow[r] \arrow[d, tail] & X[1] \\
X \arrow[r, tail] & A \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] & \\
 & W \arrow[r, equal] & W &
\end{tikzcd}.$$

But $X \in \iota(\mathcal{C})$ as well, so there exists a conflation $X \rightarrowtail C' \twoheadrightarrow W'$ with $C' \in \mathcal{C}$. Using the octahedral axiom once more, we can construct the commutative diagram

$$\begin{tikzcd}
C[-1] \arrow[r] \arrow[d, equal] & X \arrow[r, tail] \arrow[d, tail] & A \arrow[r, two heads] \arrow[d, tail] & C \arrow[d, equal] \\
C[-1] \arrow[r] & C' \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \\
 & W' \arrow[r, equal] & W' &
\end{tikzcd}$$

and since $\mathcal{C}$ is closed under extensions, $B \in \mathcal{C}$. Composing the inflations $X' \rightarrowtail A \rightarrowtail B$, we get that $X' \in \iota(\mathcal{C})$. ☐

Remark 3.55. Both the map β defined in Proposition 3.52 as well as the map ι in Proposition 3.54 can be thought as dual to the maps first proposed by C. Ingalls and H. Thomas in [IT09] between wide subcategories and torsion classes that we revisited in Section 3.1.1. The definition of β is inspired by the map α in Proposition 3.13. As we have seen, in [IT09] the map taking a wide subcategory $\mathcal{W}$ of an hereditary algebra to a torsion class was defined as $\operatorname{Fac}(\mathcal{W})$ (Proposition 3.14). In order for this map to be defined for any algebra, F. Marks and J. Šťovíček modified it to $\operatorname{Filt}(\operatorname{Fac}(\mathcal{W}))$ (Proposition 3.15). Our definition of ι recalls C. Ingalls and H. Thomas original map, and the arguments used in Proposition 3.54 to prove that ι is well-defined rely on the fact $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is a hereditary extriangulated category.

Proposition 3.56. *Let $\mathcal{C}$ be an extension-closed subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then* *i)* $\left({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1}\right)$ *is a cotorsion pair;*

ii) $\iota(\mathcal{C})^{\perp_1} = \mathcal{C}^{\perp_1}$.

Proof. Let $\mathcal{C} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a subcategory. Then $\mathcal{C} \subset {}^{\perp_1}(\mathcal{C}^{\perp_1})$, which implies that $({}^{\perp_1}(\mathcal{C}^{\perp_1})^{\perp_1} \subset \mathcal{C}^{\perp_1}$. But $\mathcal{C}^{\perp_1} \subset ({}^{\perp_1}(\mathcal{C}^{\perp_1}))^{\perp_1}$, thus

$$({}^{\perp_1}(\mathcal{C}^{\perp_1}))^{\perp_1} = \mathcal{C}^{\perp_1}.$$

In particular, $\left({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1}\right)$ is a cotorsion pair in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

For *ii*), clearly $\iota(\mathcal{C})^{\perp_1} \subset \mathcal{C}^{\perp_1}$. Let $X \in \mathcal{C}^{\perp_1}$ and $Y \in \iota(\mathcal{C})$. By definition, there exists $C \in \mathcal{C}$ and a conflation $Y \rightarrowtail C \twoheadrightarrow D$ in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. By applying $\mathbb{E}(-,X)$ and since $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is hereditary, we get an exact sequence

$$\mathbb{E}(D,X) \to \mathbb{E}(C,X) \to \mathbb{E}(Y,X) \to 0.$$

Given that $\mathbb{E}(C,X) = 0$, we have that $\mathbb{E}(Y,X) = 0$ and $X \in \iota(\mathcal{C})^{\perp_1}$, which gives the result.

□

Lemma 3.57. *Let $\mathcal{X}$ be a contravariantly resolving subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then*

$$\iota(\mathcal{X}) \subset \mathcal{X}.$$

Proof. Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and let $X \in \iota(\mathcal{X})$. Take a conflation $X \rightarrowtail T \twoheadrightarrow Y$ such that $T \in \mathcal{X}$. Since $\mathcal{X}$ is contravariantly finite, by Lemma 1.34 there exists $X' \in \mathcal{X}$, $Y' \in \mathcal{X}^{\perp_1}$ and a conflation $Y' \rightarrowtail X' \twoheadrightarrow X$ such that the corresponding deflation $X' \twoheadrightarrow X$ is a minimal right $\mathcal{X}$-approximation. By the fact that $\operatorname{Hom}_b(Y[-1], Y'[1]) = 0$ and using the octahedral axiom, we can find $C \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that the following diagram is commutative

$$\begin{tikzcd}
Y[-1] \arrow[r] \arrow[d, equal] & X' \arrow[r, tail] \arrow[d, two heads] & C \arrow[r, two heads] \arrow[d] & Y \arrow[d, equal] \\
Y[-1] \arrow[r] \arrow[dr, dashed, "0"'] & X \arrow[r, tail] \arrow[d] & T \arrow[r, two heads] \arrow[d] & Y \\
 & Y'[1] \arrow[r, equal] & Y'[1]. &
\end{tikzcd}$$

But $\mathbb{E}(T, Y') = 0$, since $T \in \mathcal{X}$ and $Y' \in \mathcal{X}^{\perp_1}$. In particular, $C \simeq T \oplus Y'$. This implies that $X' \simeq X \oplus Y'$, and therefore $X \in \mathcal{X}$ because $\mathcal{X}$ is closed under direct summands. □

Lemma 3.58. *Let $\mathcal{X}$ be a contravariantly finite resolving category of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, then*

$$\iota(\beta(\mathcal{X})) = \mathcal{X}.$$

Proof. Since $\beta(\mathcal{X}) \subset \mathcal{X}$, the previous lemma shows that $\iota(\beta(\mathcal{X})) \subset \iota(\mathcal{X}) \subset \mathcal{X}$. Consider now $U_{\mathcal{X}}$ as in Corollary 3.48. We will show that $U_{\mathcal{X}} \in \beta(\mathcal{X})$. Let $U_{\mathcal{X}} \overset{x}{\rightarrowtail} X \twoheadrightarrow X'$ be a conflation with $X \in \mathcal{X}$. By the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, there exists $W \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and a commutative diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d, equal] & U_{\mathcal{Y}} \arrow[r, tail] \arrow[d, tail] & U_{\mathcal{X}} \arrow[r, two heads, "\pi_{\mathcal{X}}"] \arrow[d, "x"] & \Lambda[1] \arrow[d, equal] \\
\Lambda \arrow[r] & W \arrow[r, tail] \arrow[d] & X \arrow[r, two heads, "\pi'_{\mathcal{X}}"] \arrow[d] & \Lambda[1] \\
& X' \arrow[r, equal] & X' &
\end{tikzcd}$$

such that the second row is a conflation. Since $\pi_{\mathcal{X}}$ is a minimal right $\mathcal{X}$-approximation, there exists $x' : X \to U_{\mathcal{X}}$ such that $\pi'_{\mathcal{X}} = \pi_{\mathcal{X}} \circ x'$, which implies that $\pi_{\mathcal{X}} \circ (x' \circ x) = \pi'_{\mathcal{X}} \circ x = \pi_{\mathcal{X}}$. Since $\pi_{\mathcal{X}}$ is minimal, we get that $x' \circ x$ is an isomorphism. In particular, x is a section, which implies that X' is a direct summand of $X \in \mathcal{X}$. This gives that $X' \in \mathcal{X}$ and $U_{\mathcal{X}} \in \beta(\mathcal{X})$.

Since we have an inflation $U_{\mathcal{Y}} \rightarrowtail U_{\mathcal{X}}$ (Corollary 3.48), $U_{\mathcal{Y}} \in \iota(\beta(\mathcal{X}))$ and $\operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}}) \subset \iota(\beta(\mathcal{X}))$. But $\iota(\beta(\mathcal{X}))$ is closed under cocones, so by Proposition 3.39, $\mathcal{X} = \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})^{\vee} \subset \iota(\beta(\mathcal{X}))$. □

Lemma 3.58 tells us that, when restricted to contravariantly finite resolving categories, the map β is injective. The following results will allow us to explicitly describe the image of β.

Proposition 3.59. *Let* $(\mathcal{X}, \mathcal{Y})$ *be a complete cotorsion pair in* $\mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$*, then*

$$\mathcal{X} = \operatorname{Cocone}(\mathcal{X} \cap \mathcal{Y}, \mathcal{X} \cap \mathcal{Y}).$$

Proof. Recall that $\mathcal{U} = \mathcal{X} \cap \mathcal{Y} = \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})$ and that $\mathcal{X} = \mathcal{U}^{\vee}$ by Corollary 3.48. Let $X \in \mathcal{X}$, there must exist $m \in \mathbb{Z}_{\geq 0}$ such that $X \in \mathcal{U}^{\vee}_{m}$, that is, we can find conflations

$$X \rightarrowtail U_0 \twoheadrightarrow X_1 \dashrightarrow X[1] \tag{3.2.3}$$

$$X_1 \rightarrowtail U_1 \twoheadrightarrow X_2 \dashrightarrow X_1[1] \tag{3.2.4}$$

with $X_i \in \mathcal{U}^{\vee}_{m-i} \subset \mathcal{X}$ for $i = 1, 2$, and $U_0, U_1 \in \mathcal{U}$ (Definition 3.34). Shifting and rotating triangles (3.2.3) and (3.2.4), and using that $\operatorname{Hom}_b(X_2, X[2]) = 0$, as well as the octahedral axiom in $\mathcal{K}^b(\operatorname{proj} \Lambda)$, we get a commutative diagram

$$\begin{tikzcd}
X_2 \arrow[r, equal] \arrow[d] & X_2 \arrow[d, "0"] & \\
X_1[1] \arrow[r] \arrow[d] & X[2] \arrow[r] \arrow[d] & U_0[2] \arrow[d, equal] \\
U_1[1] \arrow[r] & X_2[1] \oplus X[2] \arrow[r] & U_0[2]
\end{tikzcd}$$

where the last row is a triangle. Then $U_0 \to U_1 \to X_2 \oplus X[1] \dashrightarrow U_0[1]$ is a triangle as well. Since $U_0 \in \mathcal{Y}$ then $\mathbb{E}(X_2, U_0) = \operatorname{Hom}_b(X_2, U_0[1]) = 0$, and hence the morphism $X_2 \oplus X[1] \dashrightarrow U_0[1]$ must be of the form $X_2 \oplus X[1] \xrightarrow{(0,f)} U_0[1]$. This in turn implies that $U_1 \simeq X_2 \oplus \operatorname{Cone}(f)[-1]$, thus $U' = \operatorname{Cone}(f[-1])$ belongs to $\mathcal{U}$ since $\mathcal{U}$ is closed under direct summands. Remark that, by the commutativity of the previous diagram, $f[-1]$ is exactly the inflation $X \rightarrowtail U_0$. We get that $U' \simeq X_1$, and so, $X \in \operatorname{Cocone}(\mathcal{U}, \mathcal{U})$. □

Remark 3.60. For a general hereditary extriangulated category $\mathcal{K}$ Proposition 3.59 always holds. Indeed, Let $\mathcal{U} \subset \mathcal{K}$ be a presilting subcategory. Then $\mathcal{U}^{\vee}_m = \operatorname{Cocone}(\mathcal{U},\mathcal{U})$ for any $m > 2$. It suffices to show it for $m = 3$. Let $X \in \mathcal{U}^{\vee}_m = \operatorname{Cocone}(\operatorname{Cocone}(\mathcal{U},\mathcal{U}),\mathcal{U})$, then there exist conflations $X' \rightarrowtail U \twoheadrightarrow X$ and $U'' \rightarrowtail U' \twoheadrightarrow X'$ where $U, U', U'' \in \mathcal{U}$. Since $\mathcal{K}$ is hereditary, by Proposition 1.29 iv) there exists $V \in \mathcal{K}$ and a commutative diagram

$$\begin{tikzcd}
U'' \arrow[r, tail] \arrow[d, equal] & U' \arrow[r, two heads] \arrow[d, tail] & X' \arrow[d, tail] \\
U'' \arrow[r, tail] & V \arrow[r, two heads] \arrow[d, two heads] & U \arrow[d, two heads] \\
& X \arrow[r, equal] & X
\end{tikzcd}.$$

Since $\mathcal{U}$ is presilting, $V \in \mathcal{U}$ and thus $X \in \operatorname{Cocone}(\mathcal{U},\mathcal{U})$. We conclude that $\mathcal{U}^{\vee} = \operatorname{Cocone}(\mathcal{U},\mathcal{U})$. A similar argument shows that $\mathcal{U}^{\wedge} = \operatorname{Cone}(\mathcal{U},\mathcal{U})$.

Lemma 3.61. *Let $(\mathcal{X},\mathcal{Y})$ be a complete cotorsion pair in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and consider the conflation $U_{\mathcal{Y}} \rightarrowtail U_{\mathcal{X}} \twoheadrightarrow \Lambda[1]$ as in Corollary 3.48. Then*

$$\beta(\mathcal{X}) \cap \mathcal{Y} = \operatorname{add}(U_{\mathcal{X}}).$$

Proof. By the proof of Lemma 3.58 and 3.48, we know that $U_{\mathcal{X}} \in \beta(\mathcal{X}) \cap \mathcal{Y}$. Since both $\beta(\mathcal{X})$ and $\mathcal{Y}$ are additive subcategories, we get that $\operatorname{add} U_{\mathcal{X}} \subset \beta(\mathcal{X}) \cap \mathcal{Y}$. Now take $Y \in \beta(\mathcal{X}) \cap \mathcal{Y} \subset \mathcal{X} \cap \mathcal{Y} = \operatorname{add}(U_{\mathcal{X}} \oplus U_{\mathcal{Y}})$ and suppose that Y is indecomposable. If $Y \in \operatorname{add}(U_{\mathcal{X}})$, we are done. Suppose then that Y is a direct summand of $U_{\mathcal{Y}} = Y \oplus Y'$. Using the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, we have the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, tail] & Y \arrow[d, tail] & \\
U_{\mathcal{Y}} \arrow[r, tail] \arrow[d, two heads] & U_{\mathcal{X}} \arrow[r, two heads] \arrow[d, two heads] & \Lambda[1] \arrow[d, equal] \\
Y' \arrow[r, tail] & C \arrow[r, two heads] & \Lambda[1]
\end{tikzcd}$$

Since $Y \in \beta(\mathcal{X})$, the complex C must be in $\mathcal{X}$, which implies that $\mathbb{E}(C, Y) = 0$. That is, $U_{\mathcal{X}} \simeq Y \oplus C$ and $Y \in \operatorname{add}(U_{\mathcal{X}}) \cap \operatorname{add}(U_{\mathcal{Y}})$. But $U_{\mathcal{X}}$ and $U_{\mathcal{Y}}$ share no non-zero direct summands [AI12, Lemma 2.25], meaning that $Y \simeq 0$. Thus $\beta(\mathcal{X}) \cap \mathcal{Y} = \operatorname{add}(U_{\mathcal{X}})$. $\square$

Lemma 3.62. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and let $U_{\mathcal{X}}$ be as in Corollary 3.48. Then*

$$\beta(\mathcal{X}) = \operatorname{Cocone}(\operatorname{add}(U_{\mathcal{X}}), \operatorname{add}(U_{\mathcal{X}})) = \operatorname{thick}(U_{\mathcal{X}}).$$

In particular, $\beta(\mathcal{X})$ is a thick subcategory with enough injectives. All the injectives objects of $\beta(\mathcal{X})$ lie in $\operatorname{add} U_{\mathcal{X}}$ and all objects in $\beta(\mathcal{X})$ have injective dimension ≤ 1.

Proof. Let $\mathcal{U}_{\mathcal{X}} = \operatorname{add}(U_{\mathcal{X}})$ and take $U, U' \in \mathcal{U}_{\mathcal{X}} \subset \mathcal{Y}$. For every conflation $U \rightarrowtail U' \twoheadrightarrow U''$, we have that $U'' \in \mathcal{Y}$ since $\mathcal{Y}$ is closed under cones. Moreover, $U, U' \in \beta(\mathcal{X})$ which is thick, so $U'' \in \beta(\mathcal{X}) \cap \mathcal{Y}$, the later being equal to $\mathcal{U}_{\mathcal{X}}$ by Lemma 3.61. Thus,

$\mathcal{U}_{\mathcal{X}}$ is a presilting subcategory that is closed under cones. By Proposition 3.39 we get that

$$\operatorname{thick}(U_{\mathcal{X}}) = \mathcal{U}_{\mathcal{X}}^{\vee}.$$

On the other hand, Proposition 3.59 tells us that $\mathcal{X} = \operatorname{Cocone}(\mathcal{U},\mathcal{U})$. So for every $X \in \beta(\mathcal{X}) \subset \mathcal{X}$ there exists a conflation

$$X \rightarrowtail U_0 \twoheadrightarrow U_1$$

where $U_i \in \mathcal{U}$ for $i = 0, 1$. We know that there exists $U_0^{\mathcal{X}} \in \mathcal{U}_{\mathcal{X}}$ and $U_0^{\mathcal{Y}} \in \mathcal{U}_{\mathcal{Y}}$ such that $U_0 \simeq U_0^{\mathcal{X}} \oplus U_0^{\mathcal{Y}}$. Since $U_0^{\mathcal{Y}}$ is in $\operatorname{add}(\mathcal{U}_{\mathcal{Y}})$, there exists $V \in \operatorname{add}(\mathcal{U}_{\mathcal{Y}})$ and $m \in \mathbb{Z}_{\geq 0}$ such that $U_0^{\mathcal{Y}} \oplus V \simeq U_{\mathcal{Y}}^{\oplus m}$. We get a conflation $X \rightarrowtail U_0^{\mathcal{X}} \oplus U_{\mathcal{Y}}^{\oplus m} \twoheadrightarrow U_1 \oplus V$. Applying the octahedral axiom in $\mathcal{K}^b(\operatorname{proj}\Lambda)$, we get the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & U_0^{\mathcal{X}} \oplus U_{\mathcal{Y}}^{\oplus m} \arrow[r] \arrow[d, tail, "\begin{pmatrix} 1_{U_0^{\mathcal{X}}} & 0 \\ 0 & u^{\oplus m} \end{pmatrix}"] & U_1 \oplus V \arrow[d, tail] \\
X \arrow[r, tail] & U_0^{\mathcal{X}} \oplus U_{\mathcal{X}}^{\oplus m} \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\
 & \Lambda[1]^{\oplus m} \arrow[r, equal] & \Lambda[1]^{\oplus m}
\end{tikzcd}$$

Given that $X, U_0^{\mathcal{X}} \oplus U_{\mathcal{X}}^{\oplus m} \in \beta(X)$, the complex C must lie in $\beta(\mathcal{X})$, since it is a thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Moreover, $C \in \mathcal{Y}$, because $\mathcal{Y}$ is closed under extensions and contains U_1, V and $\Lambda[1]$. This implies that $C \in \beta(\mathcal{X}) \cap \mathcal{Y} = \mathcal{U}_{\mathcal{X}}$. In particular, $X \in \operatorname{Cocone}(\mathcal{U}_{\mathcal{X}},\mathcal{U}_{\mathcal{X}})$. We conclude that $\beta(\mathcal{X}) \subset \operatorname{Cocone}(\mathcal{U}_{\mathcal{X}},\mathcal{U}_{\mathcal{X}}) \subset \mathcal{U}_{\mathcal{X}}^{\vee} = \operatorname{thick}(U_{\mathcal{X}})$. Since $\operatorname{thick}(U_{\mathcal{X}})$ is the smallest thick subcategory containing $U_{\mathcal{X}}$, $\beta(\mathcal{X}) = \operatorname{Cocone}(\mathcal{U}_{\mathcal{X}},\mathcal{U}_{\mathcal{X}}) = \operatorname{thick}(U_{\mathcal{X}})$. We now show that any $U \in \mathcal{U}_{\mathcal{X}}$ is an injective object in $\beta(\mathcal{X})$. Consider a conflation $U \rightarrowtail Y \twoheadrightarrow X$ with $Y, X \in \beta(\mathcal{X})$, then there must exist a conflation $X \rightarrowtail U' \twoheadrightarrow U''$ with $U', U'' \in \mathcal{U}_{\mathcal{X}}$. We can find $A \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that the follow diagram

$$\begin{tikzcd}
U \arrow[r, tail, "f"] \arrow[d, equal] & Y \arrow[r, two heads] \arrow[d, tail, "g"] & X \arrow[d, tail] \\
U \arrow[r, tail, "f'"] & A \arrow[r, two heads] \arrow[d, two heads] & U' \arrow[d, two heads] \\
 & X' \arrow[r, equal] & X'
\end{tikzcd} .$$

commutes. Since $\mathbb{E}(U, U') = 0$, the second row splits and $A \in \mathcal{U}_{\mathcal{X}}$. That is, there exists $h : A \to U$ such that $h \circ f' = 1_U$, which in turn implies that $(h \circ g) \circ f = h \circ (f \circ g) = h \circ f' = 1_U$. We conclude that f is a section, so $U \rightarrowtail Y \twoheadrightarrow X$ splits, and U must be injective. That all injective objects are in $\mathcal{U}_{\mathcal{X}}$ follows directly from the fact that $\beta(\mathcal{X}) = \operatorname{Cocone}(\mathcal{U}_{\mathcal{X}},\mathcal{U}_{\mathcal{X}})$. This finishes the proof. □

Remark 3.63. This proposition mirrors Remark 3.18, where we showed that when Λ is a hereditary finite-dimensional $\mathbb{k}$-algebra and (M, P) a support τ-tilting pair, $\alpha(\operatorname{Fac}(M))$ is generated by the projective object M_ρ. In Lemma 3.62, the characterization of $\beta(\mathcal{X})$ in terms of the object $U_{\mathcal{X}}$ applies to any finite-dimensional $\mathbb{k}$-algebra Λ. Furthermore, we will see that $U_{\mathcal{X}}$ is injective in $\beta(\mathcal{X})$.

Proposition 3.64. *Let $\mathcal{T} \subset \mathcal{K}$ be a thick subcategory of a hereditary extriangulated category. Then, $\mathcal{T}$ has enough injectives if and only if there exists a presilting subcategory $\mathcal{U} \subset \mathcal{K}$ such that $\mathcal{U}$ is closed under cones and $\mathcal{T} = \operatorname{thick}(\mathcal{U})$.*

Proof. Suppose $\mathcal{T}$ has enough injectives and let $\mathcal{U} = \operatorname{inj}\mathcal{T}$, then $\mathcal{T} = \mathcal{U}^{\vee}$. Since $\mathbb{E}(\mathcal{T},\mathcal{U}) = 0$, in particular we have that $\mathbb{E}(\mathcal{U},\mathcal{U}) = 0$, so $\mathcal{U}$ is presilting. For any conflation $U \rightarrowtail U' \to X$ with $U, U' \in \mathcal{U}$, we must have that $X \in \mathcal{T}$ since $\mathcal{T}$ is thick. Moreover, U is injective, so the conflation must split and $X \in \mathcal{U}$, which in turn implies that $\mathcal{U}$ is closed under cones. We conclude that $\mathcal{T} = \mathcal{U}^{\vee} = \operatorname{thick}(\mathcal{U})$ by Proposition 3.39.

Conversely, suppose that $\mathcal{T} = \operatorname{thick}(\mathcal{U})$, where $\mathcal{U}$ is presilting and closed under cones. To prove the result, it suffices to show that every $U \in \mathcal{U}$ is injective. Indeed, since $\mathcal{U}$ is closed under cones, we have that $\mathcal{U}^{\vee} = \operatorname{thick}(\mathcal{U}) = \mathcal{T}$, so any object in $\mathcal{T}$ can be approximated by objects in $\mathcal{U}$. Let $U \in \mathcal{U}$ and take a conflation $U \rightarrowtail X \twoheadrightarrow Y$. Since $\mathcal{T} = \mathcal{U}^{\vee}$, there exist a conflation $X \rightarrowtail U' \twoheadrightarrow X'$ with $U' \in \mathcal{U}$ and $X' \in \mathcal{T}$. Then, there exist $A \in \mathcal{T}$ and a commutative diagram

$$\begin{tikzcd}
U \arrow[r, tail, "f"] \arrow[d, equal] & X \arrow[r, two heads] \arrow[d, tail, "g"] & Y \arrow[d, tail] \\
U \arrow[r, tail, "f'"] & U' \arrow[r, two heads] \arrow[d, two heads] & A \arrow[d, two heads] \\
& U'' \arrow[r, equal] & U''
\end{tikzcd} .$$

where the second row is a conflation. But $\mathcal{U}$ is closed under cones, so $A \in \mathcal{U}$. Moreover, $\mathcal{U}$ is presilting, so the second row must split, which implies that $U \rightarrowtail X \twoheadrightarrow Y$ does as well. We conclude that U is injective. $\square$

Lemma 3.65. *Let $\mathcal{T}$ be a thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with enough injectives, then*

$$\beta(\iota(\mathcal{T})) = \mathcal{T}.$$

Proof. Let $\mathcal{T}$ be a thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with enough injectives, by Proposition 3.64 we know that there exists a basic presilting object U such that $\mathcal{U} = \operatorname{add}(U)$ is closed under cones and $\mathcal{T} = \operatorname{thick}(\mathcal{U})$. Consider now its Bongartz completion [AI12] $\overline{U} = U' \oplus V$ given by the conflation

$$V_0 \rightarrowtail U_0 \twoheadrightarrow \Lambda[1] \tag{3.2.5}$$

where the deflation $U_0 \twoheadrightarrow \Lambda[1]$ is a minimal right U-approximation of $\Lambda[1]$, $\operatorname{add} U_0 = \operatorname{add} U' \subset \operatorname{add} U$ and $\operatorname{add} V_0 = \operatorname{add} V$ with V and U' basic. Recall that $\overline{U}$ is silting and that U is a direct summand of $\overline{U}$. By construction, $U_0 \twoheadrightarrow \Lambda[1]$ is also a minimal right $\overline{U}$-approximation and $\operatorname{add} U' \cap \operatorname{add} V = \{0\}$. Now let $W \in \operatorname{add} U/\operatorname{add} U'$, such that W is indecomposable. Since $W \in \operatorname{add}\overline{U} = \operatorname{add} U' \sqcup \operatorname{add} V$, there exists $W' \in \operatorname{add}\overline{U}$ such that $V_0 = W \oplus W'$. We can find $A \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and a commutative diagram

$$\begin{tikzcd}
W \arrow[r, tail] \arrow[d, equal] & V_0 \arrow[r, two heads] \arrow[d, tail] & W' \arrow[d, tail] \\
W \arrow[r, tail] & U_0 \arrow[r, two heads] \arrow[d, two heads] & A \arrow[d, two heads] \\
 & \Lambda[1] \arrow[r, equal] & \Lambda[1]
\end{tikzcd} .$$

such that the second row is a conflation. But W and U_0 lie in $\operatorname{add} U$ wich is closed under cones, so $A \in \operatorname{add} U$. Since U is presilting, the second row must split, in particular $W \in \operatorname{add} U' \cap \operatorname{add} V = \{0\}$. We conclude that $\operatorname{add} U' = \operatorname{add} U$.

Now, let $\mathcal{X} = (\operatorname{add} \overline{U})^{\vee}$, by Theorem 3.43 and Remark 3.49, we know that $(\mathcal{X}, \mathcal{X}^{\perp_1})$ is a cotorsion pair and that the deflation in the conflation (3.2.5) is a minimal right $\mathcal{X}$-approximation of $\Lambda[1]$. Since $\operatorname{add} U_0 = \operatorname{add} U$, we have that

$$\beta(\mathcal{X}) = \operatorname{Cocone}(U, U) = \operatorname{thick}(U) = \mathcal{T}. \tag{3.2.6}$$

Finally, we know that $\iota(\mathcal{T})$ is resolving by Proposition 3.54, that is closed under extensions, direct summands and cocones. Since $V \in \iota(\mathcal{T})$, then $\mathcal{X} = (\operatorname{add} \overline{U})^{\vee} \subset \iota(\mathcal{T})$. Moreover, $\mathcal{T} = \operatorname{Cocone}(U, U) \subset (\operatorname{add} \overline{U})^{\vee}$ and both subcategories are extension-closed, so Proposition 3.54 and Lemma 3.57 imply that $\iota(\mathcal{T}) \subset \iota((\operatorname{add} \overline{U})^{\vee}) = \iota(\mathcal{X}) \subset \mathcal{X}$. This implies that

$$\mathcal{X} = (\operatorname{add} \overline{U})^{\vee} = \iota(\mathcal{T}) \tag{3.2.7}$$

Putting (3.2.6) and (3.2.7) together, we get that

$$\beta(\iota(\mathcal{T})) = \beta(\mathcal{X}) = \mathcal{T}$$

which gives the result. □

We denote by $\text{inj-thick}\,\Lambda$ the set of thick subcategories with enough injectives of $\mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$. We are now ready to prove Theorem 3.51.

Proof of Theorem 3.51. Let $\mathcal{T} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$ be a thick subcategory, we define by $I(\mathcal{T}) = \left({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1}\right)$, which is a cotorsion pair by Proposition 3.56. For a cotorsion pair $(\mathcal{X}, \mathcal{Y})$, we let $\beta((\mathcal{X}, \mathcal{Y})) = \beta(\mathcal{X})$, which is a thick subcategory by Proposition 3.52. Thus we get well defined maps

$$\begin{tikzcd}
\operatorname{cotor} \Lambda \arrow[r, shift left, "\beta"] & \operatorname{thick} \Lambda . \arrow[l, shift left, "I"]
\end{tikzcd}$$

By Lemma 3.62, we get that for any $(\mathcal{X}, \mathcal{Y}) \in \text{c-cotor}\,\Lambda$, $\beta(\mathcal{X}) \in \text{inj-thick}\,\Lambda$. By 3.2.7 in Lemma 3.65, we know that for any $\mathcal{T} \in \text{inj-thick}\,\Lambda$, the subcategory $\iota(\mathcal{T})$ is a resolving contravariantly finite subcategory, which in turn implies that $(\iota(\mathcal{T}), \iota(\mathcal{T})^{\perp_1})$ is a complete cotorsion pair. By Proposition 3.56 *ii*), we get that

$$I(\mathcal{T}) = \left({}^{\perp_1}(\mathcal{T}^{\perp_1}), \mathcal{T}^{\perp_1}\right) = \left({}^{\perp_1}(\iota(\mathcal{T})^{\perp_1}), \iota(\mathcal{T})^{\perp_1}\right) = (\iota(\mathcal{T}), \iota(\mathcal{T})^{\perp_1}).$$

That β and I are bijections between $\text{c-cotor}\,\Lambda$ and $\text{inj-thick}\,\Lambda$ follows from Lemma 3.58 and Lemma 3.65. □

3.2.3 Linking thick and wide subcategories

The connections between τ-tilting theory and stability conditions have been studied by a vast number of authors in the last two decades, resulting in a direct bridge between sτ-tilting modules, torsion classes and semistable subcategories. In this section we propose a notion of semistability for objects in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ that will allow us to construct a bridge between the bijections of Theorem 3.1 and those of Theorem 3.51.

Definition 3.66 (**M-semistability**)**.** Let $M \in \operatorname{mod}\Lambda$ and $X = \begin{smallmatrix} X^{-1} \\ \downarrow x \\ X^0 \end{smallmatrix} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

We say that X is *M-semistable* if the map $\operatorname{Hom}_\Lambda(X^0, M) \xrightarrow{\operatorname{Hom}_\Lambda(x,M)} \operatorname{Hom}_\Lambda(X^{-1}, M)$ is an isomorphism of $\Bbbk$-vector spaces. In particular, since

$$\langle [X], [M] \rangle = \dim_\Bbbk(\operatorname{Hom}_\Lambda(X^0, M)) - \dim_\Bbbk(\operatorname{Hom}_\Lambda(X^{-1}, M)),$$

if X is M-semistable, then $\langle [X], [M] \rangle = 0$.

Remark 3.67. Note that Definition 3.66 does not depend on the choice of representative of X in its isomorphism class inside $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ thanks to Remark 1.28. Moreover, it coincides with Definition 2.15 introduced in Chapter 2.

Definition 3.68. Let $\mathcal{H}$ be a subcategory of $\operatorname{mod}\Lambda$. We define $\mathscr{T}(\mathcal{H})$ to be the full subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ whose objects are all complexes X such that X is N-semistable $\forall\ N \in \mathcal{H}$. Similarly, if $\mathcal{C} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, we define $\mathscr{W}(\mathcal{C}) := \bigcap_{X \in \mathcal{C}} \mathscr{W}(X)$ as the full subcategory of modules N such that all objects in $\mathcal{C}$ are N-semistable.

Remark 3.69. The wide subcategories $\mathscr{W}(\mathcal{C})$ were already considered in work of L. Angeleri Hügel, F. Marks and J. Vitória [AHMV16a, AHMV16b]. In their work, they are defined with respect to morphisms between two (not necessarily finite-dimensional) projective Λ-modules, and are key to their generalization of large tilting modules and support τ-tilting modules in the infinite-dimensional setting.

Proposition 3.70. *Let Λ be a finite-dimensional $\Bbbk$- algebra, then*

1. $\forall\ \mathcal{H} \subset \operatorname{mod}\Lambda$, $\mathscr{T}(\mathcal{H}) \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *is a thick subcategory.*

2. $\forall\ \mathcal{C} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, $\mathscr{W}(\mathcal{C}) \subset \operatorname{mod}\Lambda$ *is a wide subcategory.*

3. *For any subcategories $\mathcal{H} \subset \operatorname{mod}\Lambda$ and $\mathcal{C} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$,*

$$\mathcal{H} \subset \mathscr{W}(\mathscr{T}(\mathcal{H})),$$
$$\mathcal{C} \subset \mathscr{T}(\mathscr{W}(\mathcal{C})).$$

4. *For any subcategories $\mathcal{H} \subset \operatorname{mod}\Lambda$ and $\mathcal{C} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.*

$$\mathscr{T}(\mathcal{H}) = \mathscr{T}(\operatorname{wide}(\mathcal{H}))$$
$$\mathscr{W}(\mathcal{C}) = \mathscr{W}(\operatorname{thick}(\mathcal{C}))$$

Proof. We only prove (1) for $\mathcal{H} = \{M\}$ with $M \in \operatorname{mod} \Lambda$. The result follows noting that $\mathscr{T}(\mathcal{H}) = \bigcap_{M \in \mathcal{H}} \mathscr{T}(M)$. The statement in (2) follows using similar arguments.

Closure under extensions: Let $X \rightarrowtail Y \twoheadrightarrow Z$ be a conflation and suppose X and Z are in $\mathscr{T}(M)$. As we have seen in Remark 1.28, we can find $P \in \operatorname{proj} \Lambda$ such that $X \rightarrowtail Y \oplus \begin{smallmatrix} P \\ \| \\ P \end{smallmatrix} \twoheadrightarrow Z$ is a conflation in $\mathcal{C}^{[-1,0]}(\operatorname{proj} \Lambda)$. Applying $\operatorname{Hom}_\Lambda(-, M)$ to the associated exact sequences, we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_\Lambda(Z^0, M) \arrow[r] \arrow[d, "z^*"] & \operatorname{Hom}_\Lambda(Y^0 \oplus P, M) \arrow[r] \arrow[d, "(y \oplus 1_P)^*"] & \operatorname{Hom}_\Lambda(X^0, M) \arrow[r] \arrow[d, "x^*"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(Z^{-1}, M) \arrow[r] & \operatorname{Hom}_\Lambda(Y^{-1} \oplus P, M) \arrow[r] & \operatorname{Hom}_\Lambda(X^{-1}, M) \arrow[r] & 0
\end{tikzcd} \tag{3.2.8}$$

where z^* and x^* are isomorphisms by hypothesis. Since $(y \oplus 1_P)^* = y^* \oplus 1_{\operatorname{Hom}_\Lambda(P,M)}$, y^* is an isomorphism as well.

Closure under cones and cocones: Take a conflation as before and suppose that X and Y are M-semistable. In particular, $0 = \langle [Y], [M] \rangle = \langle [X], [M] \rangle + \langle [Z], [M] \rangle = 0 + \langle [Z], [M] \rangle$. As before, there exists $P \in \operatorname{proj} \Lambda$ and a commutative diagram like (3.2.8). Since $(y \oplus 1_P)^*$ and x^* are isomorphisms we can deduce that z^* is injective. Moreover, z^* is a linear map between vector spaces of same dimension, so it must be bijective. The proof of $\mathscr{T}(M)$ being closed under cocones is dual.

Closure under direct summands: Let $X \in \mathscr{T}(M)$ such that $X \simeq X' \oplus X''$. Since we have inflations $X' \rightarrowtail X$ and $X'' \rightarrowtail X$, then $\langle [X'], [M] \rangle, \langle [X''], [M] \rangle \geq 0$ (see Proposition 2.22 for more details). But $0 = \langle [X], [M] \rangle = \langle [X'], [M] \rangle + \langle [X''], [M] \rangle$, so both terms must be equal to 0. Take now the conflation $X'' \rightarrowtail X \twoheadrightarrow X'$ and $P \in \operatorname{proj} \Lambda$ such that we have a commutative diagram like (3.2.8). This time around, $x^* \oplus 1_P^*$ is an isomorphism, which implies that $(x')^*$ is bijective since it is injective and $\langle [X'], [M] \rangle = 0$.

We proceed to prove (3). Since for any subcategory $\mathcal{H} \subset \operatorname{mod} \Lambda$, $\mathscr{T}(\mathcal{H}) = \bigcap_{M \in \mathcal{H}} \mathscr{T}(M)$, the map $\mathscr{T}$ reverses inclusions. Take $M \in \mathcal{H}$, then all $X \in \mathscr{T}(\mathcal{H})$ satisfy that X is M-semistable, that is, $M \in \mathscr{W}(\mathscr{T}(\mathcal{H}))$. The rest of the statement follows from similar arguments.

Lastly, consider $\mathcal{C} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$. Since $\mathscr{W}$ reverses inclusions and $\mathcal{C} \subset \operatorname{thick}(\mathcal{C})$, then $\mathscr{W}(\operatorname{thick}(\mathcal{C})) \subset \mathscr{W}(\mathcal{C})$. We have seen that $\mathscr{T}(\mathscr{W}(\mathcal{C}))$ is a thick subcategory that contains $\mathcal{C}$, so $\operatorname{thick}(\mathcal{C}) \subset \mathscr{T}(\mathscr{W}(\mathcal{C}))$. Applying (3) to $\mathcal{H} = \mathscr{W}(C)$, we have the following inclusions :

$$\mathscr{W}(\mathcal{C}) \subset \mathscr{W}(\mathscr{T}(\mathscr{W}(\mathcal{C}))) \subset \mathscr{W}(\operatorname{thick}(\mathcal{C})) \subset \mathscr{W}(\mathcal{C}).$$

Thus $\mathscr{W}(\mathcal{C}) = \mathscr{W}(\operatorname{thick}(\mathcal{C}))$. That $\mathscr{T}(\mathcal{H}) = \mathscr{T}(\operatorname{wide}(\mathcal{H}))$ for any subcategory $\mathcal{H}$ of $\operatorname{mod} \Lambda$ follows from the same argument. This proves (4).

□

Remark 3.71. In Proposition 4.53 of Chapter 4, we show that the maps $\mathscr{W}$ and $\mathscr{T}$ can be defined in homological terms, which provides another proof of the statements 1 and 2 in the previous proposition.

We are now ready to state the main theorem of this chapter.

Theorem 3.72. *Let Λ be a finite-dimensional $\Bbbk$-algebra. There exist well defined maps*

$$\begin{tikzcd}
\text{wide}\,\Lambda \arrow[r, shift left, "\mathscr{T}"] & \text{thick}\,\Lambda \arrow[l, shift left, "\mathscr{W}"]
\end{tikzcd}$$

such that, when restricted to thick subcategories with enough injectives and left finite wide subcategories, they make the following diagram commute

$$\begin{tikzcd}
& & \text{2-silt}\,\Lambda \arrow[dr, "\operatorname{thick}(U_\rho)"] & & \\
& \text{c-cotor}\,\Lambda \arrow[ur, "\Psi"] \arrow[rr, "\beta"] \arrow[dddd, "\Phi"] & & \text{inj-thick}\,\Lambda \arrow[dddd, "\mathscr{W}"] & \\
\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) & & & & \\
{} \arrow[rrrr, dash, dashed] & & & & {} \\
& & & & \operatorname{mod}\Lambda \\
& \text{f-tors}\,\Lambda \arrow[rr, "\alpha"] & & \text{l-wide}\,\Lambda &
\end{tikzcd}$$

In particular, $\mathscr{W}$ *and* $U \in \text{2-silt}\,\Lambda \mapsto \operatorname{thick}(U_\rho) \in \text{inj-thick}\,\Lambda$ *are bijective.*

Proof. The first part of the statement follows from Proposition 3.70. We show that the center square of the diagram is commutative, that the upper triangle is as well follows from Lemma 3.62. Let $(\mathcal{X}, \mathcal{Y})$ be a complete cotorsion pair. We prove that $\mathscr{W}(\beta(\mathcal{X})) = \alpha(H^0(\mathcal{Y}))$. Proposition 3.47 implies that $H^0(\mathcal{Y}) = \operatorname{Fac}(H^0(U_\mathcal{X} \oplus U_\mathcal{Y}))$. By [MŠ17, Lemma 3.8] and [Yur18, Lemma 3.5] we have that

$$\alpha(\operatorname{Fac}(H^0(U_\mathcal{X} \oplus U_\mathcal{Y}))) = \mathscr{W}(U_\mathcal{X}).$$

Moreover, $\mathscr{W}(\beta(\mathcal{X})) = \mathscr{W}(\operatorname{thick}(U_\mathcal{X}))$ by Lemma 3.62, and $\mathscr{W}(U_\mathcal{X}) = \mathscr{W}(\operatorname{thick}(U_\mathcal{X}))$ by Proposition 3.70 (4). Putting all these equalities together we get that

$$\mathscr{W}(\beta(\mathcal{X})) = \mathscr{W}(\operatorname{thick}(U_\mathcal{X})) = \mathscr{W}(U_\mathcal{X}) = \alpha(\operatorname{Fac}(H^0(U_\mathcal{X} \oplus U_\mathcal{Y}))) = \alpha(H^0(\mathcal{Y}))$$

which gives us the result. ☐

Example 3.73. Let Q be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then the Auslander-Reiten quiver of $\operatorname{mod}\Bbbk Q$ is the following

$$\begin{tikzcd}
& & P_3 = I_1 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\
P_1 \arrow[ur] & & S_2 \arrow[ur] & & I_3
\end{tikzcd}.$$

All minimal projective presentations of indecomposable modules are indecomposable objects in $\mathcal{K}_{\Bbbk Q} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Bbbk Q)$, as are the objects $P \to 0$ where P is an indecomposable projective module. Then, the AR quiver of $\mathcal{K}_{\Bbbk Q}$ is given by

$$\begin{tikzcd}
& & 0 \to P_3 \arrow[dr] & & P_1 \to 0 \arrow[dr] & & \\
& 0 \to P_2 \arrow[ur] \arrow[dr] & & P_1 \xrightarrow{\beta\alpha} P_3 \arrow[ur] \arrow[dr] & & P_2 \to 0 \arrow[dr] & \\
0 \to P_1 \arrow[ur] & & P_1 \xrightarrow{\alpha} P_2 \arrow[ur] & & P_2 \xrightarrow{\beta} P_3 \arrow[ur] & & P_3 \to 0
\end{tikzcd}.$$

In Table 3.1, we show all silting objects, their respective cotorsion pairs, thick subcategories, wide subcategory and torsion class given by the bijections in Corollary 3.33. The dots correspond to the objects depicted in the AR quiver of $\mathcal{K}_{\Bbbk Q}$ (or $\operatorname{mod} \Bbbk Q$), and the shaded areas correspond to the subcategory additively generated by the dots they contain. In the second column, the blue shaded area in each figure depicts the subcategory $\mathcal{X}$, while the orange shaded area plays the role of $\mathcal{Y}$ for the cotorsion pairs $(\mathcal{X}, \mathcal{Y})$ they illustrate.

Example 3.74 ($\mathscr{W}$ is not a bijection in general). Consider now the Kronecker quiver

$$Q = \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \end{tikzcd} .$$

Let $\mathcal{C}$ be thick subcategory of $\mathcal{K}_{\Bbbk Q}$ whose objects are the projective presentations of regular modules. In particular, any object $X \in \mathcal{C}$ satisfies that $[X] = n[P_1] - n[P_2] = (n, -n)$ for some $n \in \mathbb{Z}_{>0}$. Let $M \neq 0$ be an indecomposable module in $\mathscr{W}(\mathcal{C}) \subset \operatorname{mod} \Bbbk Q$ with minimal projective resolution $X_M \in \mathcal{K}_{\Bbbk Q}$. Since $\langle [X], M \rangle = 0$ for all $X \in \mathcal{C}$, M cannot be pre-projetive or pre-injective. If M is regular, then $X_M \in \mathcal{C}$, but M cannot be X_M-semistable. We conclude that $\mathscr{W}(\mathcal{C}) = \{0\} = \mathscr{W}(\mathcal{K}_{\Bbbk Q})$, and $\mathscr{W}$ is not a bijection in general.

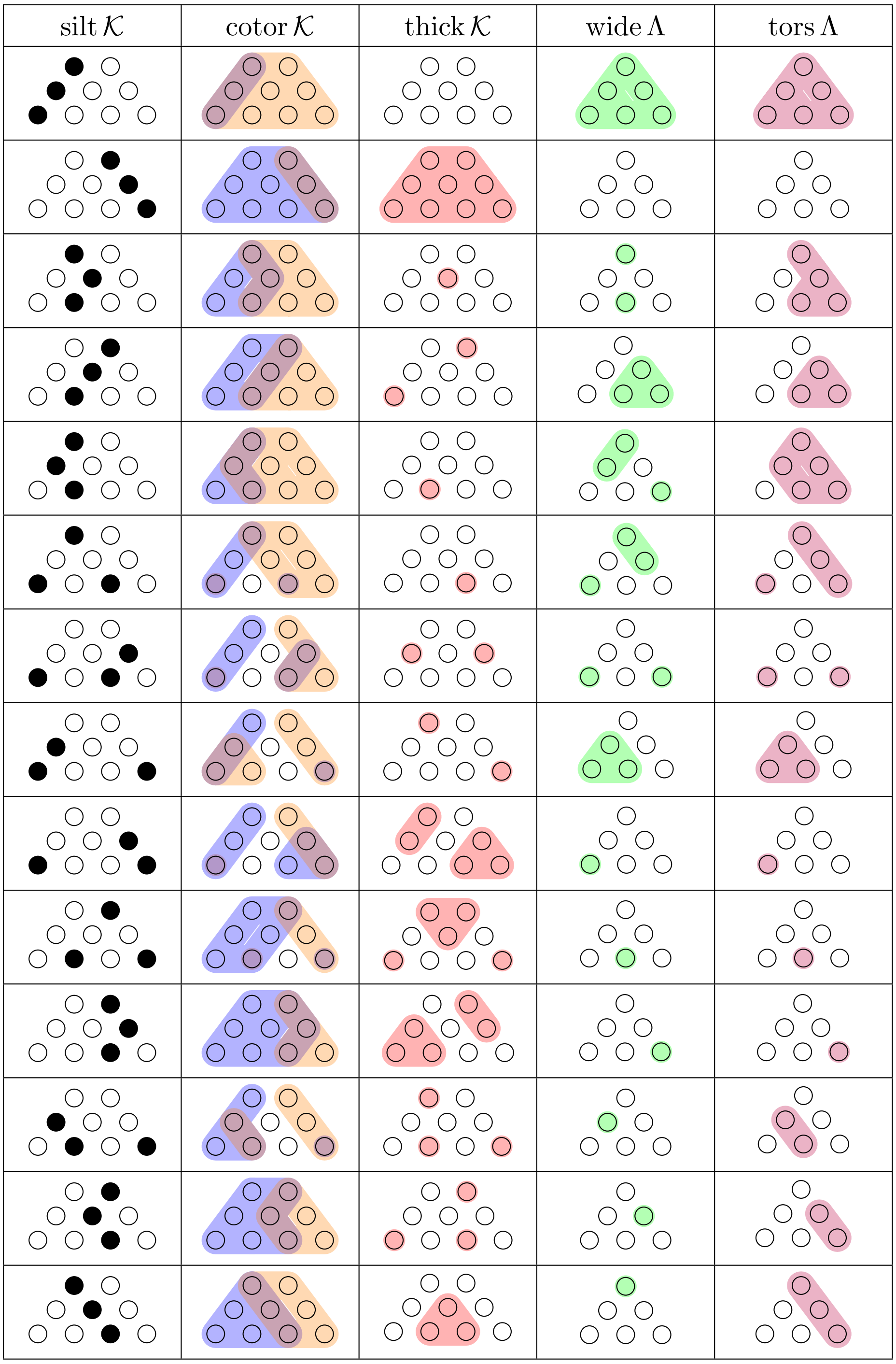


Table 3.1: Example of Corollary 3.33.

Chapter 4

g-finite algebras

The notion of support τ-tilting module [AIR14] was inspired in part by the additive categorification of cluster algebras [Ami09, BMR$^+$06]. When Λ is the Jacobi algebra $\mathcal{J}(Q,W)$ associated with a quiver Q with (non-degenerate) potential W, the set of basic 2-term silting complexes is in bijection with the set of *clusters* of the cluster algebra $\mathcal{A}_Q$ associated to (Q,W). One of the first questions to be settled when cluster algebras were introduced, is whether the set of clusters of a given cluster algebra $\mathcal{A}_Q$ is finite. This turned ou to be equivalent to being *mutation equivalent* to a quiver Q being of Dynkin type [FZ03]. Since Dynkin quiver are representation finite, $\mathcal{K}^{[-1,0]}(\operatorname{proj} J(Q,W))$ has only finitely many isoclasses of basic 2-term silting objects. For a general $\Bbbk$-algebra Λ this does not have to be the case, prompting the introduction of the following definition.

Definition 4.1. [DIJ19] Let Λ be a finite-dimensional $\Bbbk$-algebra. We say that Λ is *g-finite* if it admits only finitely many isomorphism classes of basic 2-term silting objects.

The study of g-finite algebras was introduced by L. Demonet, O. Iyama and G. Jasso in [DIJ19], who showed that an algebra Λ being g-finite has deep implications on the structure of $\operatorname{mod}\Lambda$.

Theorem 4.2. *[DIJ19, Theorems 3.8 and 4.2] Let Λ be a finite-dimensional $\Bbbk$-algebra. The following are equivalent:*

1. *Λ is g-finite.*
2. *There exist finitely many functorially finite torsion classes in* $\operatorname{mod}\Lambda$.
3. *All torsion classes in* $\operatorname{mod}\Lambda$ *are functorially finite.*
4. *There exist finitely many bricks in* $\operatorname{mod}\Lambda$.

The following corollary follows from the definition of a left finite wide subcategory and Theorem 4.2 3. and Proposition 3.16.

Corollary 4.3. *Let Λ be a finite-dimensional $\Bbbk$-algebra. If Λ is g-finite, then all wide subcategories are left finite.*

In this chapter we study the consequences that being g-finite has on the correspondences in between complete cotorsion pairs, 2-term silting objects and thick subcategories with enough injectives in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ introduced in Chapter 3. Specifically, we provide analogs of Theorem 4.2 and Corollary 4.3.

Theorem 4.4. *Let Λ be a finite-dimensional $\Bbbk$-algebra. The following are equivalent:*

1. *Λ is g-finite.*
2. *There exist finitely many complete cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.*
3. *All cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ are complete.*
4. *There exist finitely many thick subcategories in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.*

A key ingredient in the proof is the following extension of a result of D. Pauksztello and A. Zvonareva [PZ23].

Theorem (4.44). *Let Λ be a finite-dimensional $\Bbbk$-algebra. Then the functor $H^0 : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \to \operatorname{mod}\Lambda$ induces a bijection*

$$\begin{aligned} H^0 : \operatorname{cotor}\Lambda &\to \operatorname{tors}\Lambda \\ (\mathcal{X},\mathcal{Y}) &\mapsto H^0(\mathcal{Y}) \end{aligned}$$

We conclude this chapter by providing an analog to Corollary 4.3 in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Theorem (4.56). *Suppose Λ is g-finite. Then all thick subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ have enough injectives.*

In contrast to its counterpart in $\operatorname{mod}\Lambda$, the previous statement does not immediately follow from Theorem 4.2. This is because the property of a thick category $\mathcal{T}$ having enough injectives does not necessarily depend on the completeness of the associated cotorsion pair $\iota(\mathcal{T})$. To establish Theorem 4.56, we rely on the reduction of an extriangulated category with respect to a presilting object, which we review and study in Section 4.2.

4.1 DG algebras, silting objects, simple-minded collections and semibricks

The goal of this section is to introduce the necessary preliminaries for Sections 4.2 and 4.3. We introduce the extriangulated category of 2-term perfect complexes $\operatorname{per}^{[-1,0]}(\Gamma)$ over a dg algebra Γ, of which $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, studied in previous chapters, is a special case. For a broader introduction to dg categories see [Kel94, Kel06]. Most of the results in this section are taken from [BY13, KY14].

4.1.1 DG algebras and dg categories

Definition 4.5. A *differential graded algebra*, or dg algebra for short, is a graded $\Bbbk$-algebra $\Gamma = \bigoplus_{i\in\mathbb{Z}} \Gamma^i$ equipped with a homogeneous $\Bbbk$-linear map of degree one $d : \Gamma \to \Gamma$ such that

i) $d(ab) = d(a)b + (-1)^i a d(b) \quad \forall a \in \Gamma^i$ and $b \in \Gamma$;

ii) $d \circ d = 0$.

We call d the *differential* of Γ. A morphism between dg algebras $f : \Gamma \to \Theta$ is a map of graded $\Bbbk$-algebras which is compatible with the differential, that is, if for all $a \in \Gamma$

$$f(d_\Gamma(a)) = d_\Theta(f(a))$$

.

Let Λ be a finite-dimensional $\Bbbk$-algebra. Then Λ is a dg algebra when seen as a graded algebra concentrated in degree 0 with differential $d = 0$. For $i \in \mathbb{Z}$, the i-th cohomology of a dg algebra Γ is defined as

$$H^i(\Gamma) = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}.$$

A *quasi-isomorphism* of dg algebras is a dg algebra morphism $f : \Gamma \to \Theta$ that induces $\Bbbk$-algebra isomorphisms $f^i : H^i(\Gamma) \to H^i(\Theta)$.

Definition 4.6. Let Γ be a dg algebra and denote by d_Γ its differential. A dg Γ-module is a graded right Γ-module $X = \bigoplus_{i\in\mathbb{Z}} X^i$ equipped with a homogeneous $\Bbbk$-linear map of degree one $d_X : X \to X$ such that

$$d_X(xa) = d_X(x)a + (-1)^i x\, d_\Gamma(a) \ \forall a \in \Gamma \text{ and } x \in X^i.$$

For a given dg Γ-module X we denote by $H^i(X) = \operatorname{Ker} d_X^i / \operatorname{Im} d_X^{i-1}$.

For given X and Y dg Γ-modules we define the dg $\Bbbk$-module

$$\mathcal{H}\mathrm{om}_\Gamma(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathcal{H}\mathrm{om}^i_\Gamma(X,Y)$$

where $\mathcal{H}\mathrm{om}^i(X,Y)$ is the subset of $\prod\limits_{j\in\mathbb{Z}} \operatorname{Hom}_\Gamma(X^j, Y^{j+i})$ of elements $(f_j)_{j\in\mathbb{Z}}$ such that

$$f_j(x)a = f_{j+n}(xa) \text{ for all } x \in X^j \text{ and } a \in \Gamma^n;$$

whose differential is given by the map

$$f \in \mathcal{H}\mathrm{om}^i_\Gamma(X,Y) \mapsto f \circ d_X - (-1)^i d_Y \circ f.$$

Note that given two dg Γ-modules X and Y, the kernel of d^0, which we denote by $Z^0(\mathcal{H}\mathrm{om}_\Gamma(X,Y))$, is nothing but the set of Γ-linear maps that commute with the differentials of X and Y. For any dg Γ-module X with differential d_X and $i \in \mathbb{Z}$, let $X[i]$ be the dg module whose underlying graded Γ-module is $\bigoplus_{j\in\mathbb{Z}} X^{i+j}$ equipped with the differential $(-1)^i d_X$. As in Example 4.11, the sets $H^i(\mathcal{H}\mathrm{om}_\Gamma(X,Y))$ correspond to the sets $Z^0(\mathcal{H}\mathrm{om}_\Gamma(X,Y[i]))$ modulo the homotopy relation.

Definition 4.7. Let Γ be a dg algebra. We denote by $\mathcal{C}(\Gamma)$ the category whose objects are dg Γ-modules and whose morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}(\Gamma)}(X,Y) = Z^0(\mathcal{H}\mathrm{om}_\Gamma(X,Y)).$$

Denote by $\mathcal{K}(\Gamma)$ the category whose objects are the same as those of $\mathcal{C}(\Gamma)$ but whose morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{K}(\Gamma)}(X,Y) = H^0(\mathcal{H}\mathrm{om}_\Gamma(X,Y)).$$

The category $\mathcal{K}(\Gamma)$ is triangulated when equipped with the shift functor $[1] : \mathcal{K}(\Gamma) \to \mathcal{K}(\Gamma)$. Note that any map $f \in \operatorname{Hom}_{\mathcal{C}(\Gamma)}(X, Y)$ induces maps $f^i : H^i(X) \to H^i(Y)$, and that if two such maps f and g coincide in $\operatorname{Hom}_{\mathcal{K}(\Gamma)}(X, Y)$, then the maps they induce in cohomology are the same. We say that a map $f \in \operatorname{Hom}_{\mathcal{K}(\Gamma)}(X, Y)$ is a *quasi-isomorphism* if it induces isomorphisms in cohomplogy.

Definition 4.8. Let Γ be a dg algebra. The *derived category* associated to Γ is the triangulated quotient

$$\mathcal{D}(\Gamma) = \mathcal{K}(\Gamma)/\mathcal{N},$$

where $\mathcal{N}$ is the thick subcategory of objects whose cohomologies are all 0. In other words, $\mathcal{D}(\Gamma)$ is the Verdier localisation of $\mathcal{K}(\Gamma)$ by quasi-isomorphisms.

We will denote by $\mathcal{D}_{fd}(\Gamma)$ the full subcategory of $\mathcal{D}(\Gamma)$ of dg Γ-modules whose total cohomology is finite-dimensional. We will consider as well the category $\operatorname{per}(\Gamma) = \operatorname{thick}_{\mathcal{D}(\Gamma)}(\Gamma) \subset \mathcal{D}(\Gamma)$, know as the category of perfect complexes over Γ.

Example 4.9. Let Λ be a finite-dimensional $\mathbb{k}$-algebra. Seen as a dg algebra concentrated in degree 0, then $\mathcal{C}(\Lambda)$ is precisely the category $\mathcal{C}(\operatorname{Mod} \Lambda)$ of complexes of (not necessarily finite-dimensional) Λ-modules, $\mathcal{D}(\Lambda)$ is $\mathcal{D}(\operatorname{Mod} \Lambda)$, $\mathcal{D}_{fd}(\Lambda)$ corresponds to $\mathcal{D}^b(\operatorname{mod} \Lambda)$ and $\operatorname{per}(\Lambda)$ is equivalent to $\mathcal{K}^b(\operatorname{proj} \Lambda)$.

4.1.2 Non-positive dg algebras

Definition 4.10. We say a a dg algebra Γ is *non-positive* if $\Gamma^i = 0$ for all $i \geq 1$. We say that Γ it is *finite-dimensional* if it is finite-dimensional as a $\mathbb{k}$-vector space.

Example 4.11. Let Λ be a finite-dimensional $\mathbb{k}$-algebra and let $X \in \mathcal{C}^b(\operatorname{proj} \Lambda)$ with differential $(x^i)_{i \in \mathbb{Z}}$. Consider $\Gamma_X = \mathcal{H}om_\Lambda(X, X) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}om^i_\Lambda(X, X)$ where $\mathcal{H}om^i_\Lambda(X, X) = \prod\limits_{j \in \mathbb{Z}} \operatorname{Hom}_\Lambda(X^j, X^{j+i})$. Then Γ_X is a finite-dimensional dg algebra whose i-th cohomology satisfies $H^i(\Gamma_X) \simeq \operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj} \Lambda)}(X, X[i])$. Moreover, if we suppose X to be presilting in $\mathcal{K}^b(\operatorname{proj} \Lambda)$, then Γ_X is quasi-isomorphic to its dg subalgebra

$$\Gamma_X^{\leq 0} = \left(\bigoplus_{i<0} \mathcal{H}om^i_\Lambda(X, X) \right) \oplus \operatorname{Hom}_{\mathcal{C}^b(\operatorname{proj} \Lambda)}(X, X),$$

which is finite-dimensional and non-positive.

Theorem 4.12. *[KY14, Lemma 4.1] [Kel06, Theorem 3.8 b)] Let $\mathcal{C}$ be an algebraic triangulated category, that is, equivalent to the stable category of a Frobenius category. Suppose that $\mathcal{C}$ has split idempotents and an object $X \in \mathcal{C}$ such that* $\operatorname{thick}(X) = \mathcal{C}$. *Then there exists a dg algebra Γ and a triangle equivalence*

$$\operatorname{per}(\Gamma) \xrightarrow{\simeq} \mathcal{C}$$

which takes Γ to X. Moreover, if X is silting, then Γ is non-positive. In particular $H^0(\Gamma) \simeq \operatorname{Hom}_{\operatorname{per}(\Gamma)}(\Gamma, \Gamma) \simeq \operatorname{Hom}_{\mathcal{C}}(X, X)$.

4.1.3 2-term silting objects and 2-term simple-minded collections

Let Γ be a finite-dimensional non-positive dg algebra. Recall that a silting object V of $\mathcal{D}$ is an object satisfying $\mathrm{Hom}_{\mathcal{D}}(V, \Sigma^i V) = 0$ for all $i > 0$ and such that $\mathrm{thick}_{\mathcal{D}}(V) = \mathcal{D}$. Let V be a silting object in $\mathcal{D}$, we say that an object $X \in \mathcal{D}$ is 2-*term* with respect to V if there exist $Y, Y' \in \mathrm{add}(V)$ and a triangle

$$Y' \to Y \to X \dashrightarrow Y'[1].$$

We will denote by $V * \Sigma V$ the full subcategory of 2-term objects with respect to V. When $\mathcal{D} = \mathrm{per}(\Gamma)$, where Γ is a finite-dimensional non-positive dg algebra, then Γ is itself a silting object in $\mathrm{per}(\Gamma)$, since $\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Gamma[i]) \simeq H^i(\Gamma) = 0$ for all $i > 0$. We write $\mathrm{per}^{[-1,0]}(\Gamma) := \Gamma * \Gamma[1]$ for the full subcategory of 2-term objects of $\mathrm{per}\,\Gamma$ with respect to Γ. We will refer to $\mathrm{per}^{[-1,0]}(\Gamma)$ simply as the extriangulated category of 2-term objects of Γ and denote by $2\text{-silt}\,\Gamma$ the set isomorphism classes of basic 2-term silting objects in $\mathrm{per}(\Gamma)$.

Lemma 4.13 (Bongartz Completion)**.** *[IJY14, Lemma 4.2] Let $\mathcal{D}$ be a triangulated category which is essentially small, Krull–Schmidt, $\Bbbk$-linear and Hom-finite with shift functor Σ. Suppose that $\mathcal{D}$ possesses a silting object $V \in \mathcal{D}$. Let $U \in \mathcal{D}$ be a presilting object in $V * \Sigma V$, then there exists and object $U' \in V * \Sigma V$ such that $U \oplus U'$ is a silting object in $\mathcal{D}$.*

Proposition 4.14. *[IJY14, Proposition 4.3] Let $\mathcal{D}$ be as in Lemma 4.13 and let U be a presilting object in $V * \Sigma V$. Then U is silting if and only if $|V| = |U|$.*

Definition 4.15. [Ric02] Let $\mathcal{C}$ be a triangulated category with shift functor Σ. A collection of objects $X_1, X_2, \cdots, X_r$ is said to be *simple-minded* if the following conditions hold for any $1 \leq i, j \leq n$:

i) $\mathrm{Hom}_{\mathcal{C}}(X_i, \Sigma^m X_j) = 0$ for all $m < 0$,

ii) $\mathrm{End}_{\mathcal{C}}(X_i)$ is a division algebra and $\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) = 0$ when $i \neq j$.

iii) $\mathrm{thick}_{\mathcal{C}}(\{X_1, \cdots, X_r\}) = \mathcal{C}$.

We denote by $\mathrm{smc}\,\Gamma$ the set of isomorphism classes of simple-minded collections of $\mathcal{D}_{fd}(\Gamma)$.

If Γ is a finite-dimensional non-positive a dg algebra, then the set $\{S_1, \cdots, S_n\}$ of pairwise non-isomorphic simple $H^0(\Gamma)$-modules is a simple-minded collection in $\mathcal{D}_{fd}(\Gamma)$ [BY13, Appendix A.1]. We say that a simple-minded collection of $\mathcal{D}_{fd}(\Gamma)$ is 2-*term* if $H^j(X_i) = 0$ for all $j \neq 0, -1$ and all $1 \leq i \leq n$. We denote by $2\text{-smc}\,\Gamma$ the set of isomorphism classes of 2-term simple-minded collections.

Remark 4.16. [BY13, Remark 4.11] Let Λ be a finite-dimensional $\Bbbk$-algebra. Suppose that $\{X_1, \cdots, X_n\}$ is a 2-term simple-minded collection in $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$, then for any $1 \leq i \leq n$ the object X_i belongs to either $\mathrm{mod}\,\Lambda$ or $(\mathrm{mod}\,\Lambda)[1]$.

Theorem 4.17. *[KN13, KY14] [BY13, Corollary 4.1] Let Γ be a homologically smooth[1] non-positive dg algebra or a finite-dimensional $\Bbbk$-algebra. Then there exists a bijection*

$$\Omega : \mathrm{silt}\,\Gamma \to \mathrm{smc}\,\Gamma$$

---

[1] A dg algebra Γ is homologically smooth if as a dg bimodule ${}_{\Gamma}\Gamma_{\Gamma}$ over itself it has a bounded resolution by finitely generated projective dg bimodules

that restricts to a bijection $\Omega : 2\text{-silt}\,\Gamma \to 2\text{-smc}\,\Gamma$ *between the set of* 2*-term silting objects and* 2*-term simple-minded collections.*

When $\Gamma = \Lambda$ is a finite-dimensional $\mathbb{k}$-algebra, the bijection in Theorem 4.17 is given in the following way. Let U be a silting object in $\operatorname{per}\Lambda$. By Theorem 4.12, there exists a non-positive dg algebra B together with a triangle equivalence $\mathcal{D}(B) \to \mathcal{D}(\Lambda)$ that takes B to U. The simple-minded collection $\{X_1, \cdots, X_n\}$ corresponding to U under the map Ω is the image under the equivalence $\mathcal{D}(B) \to \mathcal{D}(\Lambda)$ of a complete collection of non-isomorphic simple $H^0(B)$-modules. In particular, any simple-minded collection has $n = |U| = |\Lambda|$ elements.

We recall now the relation between 2-term silting objects of a finite-dimensional non-positive dg algebra Γ and those of the finite-dimensional algebra $H^0(\Gamma)$. Let $\bar{\Gamma} = H^0(\Gamma)$ and $p : \Gamma \to \bar{\Gamma}$ the canonical projection. The map p gives rise to the triangulated functor

$$\begin{aligned} p_* : \operatorname{per}(\Gamma) &\longrightarrow \operatorname{per}(\bar{\Gamma}) \\ X &\longmapsto X \otimes_{\Gamma} \bar{\Gamma}, \end{aligned}$$

which we refer to as the *induction functor.* Since $\operatorname{Hom}_{\operatorname{per}(\Gamma)}(\Gamma, \Gamma) = \bar{\Gamma} = \operatorname{Hom}_{\bar{\Gamma}}(\bar{\Gamma}, \bar{\Gamma})$, then p_* induces an equivalence $\operatorname{add}_{\operatorname{per}(\Gamma)}(\Gamma) \simeq \operatorname{add}_{\operatorname{per}(\bar{\Gamma})}(\bar{\Gamma})$, where $\operatorname{per}(\bar{\Gamma}) \simeq \mathcal{K}^b(\operatorname{proj}\bar{\Gamma})$.

Proposition 4.18. *[BY13, Proposition A.5] Let* $\mathcal{I}$ *be the ideal of* $\operatorname{per}^{[-1,0]}(\Gamma)$ *consisting of morphisms factoring through morphisms* $X[1] \to Y$ *with* $X, Y \in \operatorname{add}_{\operatorname{per}(\Gamma)}(\Gamma)$*. Then* $\mathcal{I}^2 = 0$ *and* p_* *induces an equivalence of* $\mathbb{k}$*-linear categories* $\operatorname{per}^{[-1,0]}(\Gamma)/\mathcal{I} \to \mathcal{K}^{[-1,0]}(\operatorname{proj} H^0(\Gamma))$*. In particular,* p_* *is full, detects isomorphisms, preserves indecomposability and induces a bijection between isomorphism classes of objects of* $\operatorname{per}^{[-1,0]}(\Gamma)$ *and* $\mathcal{K}^{[-1,0]}(\operatorname{proj} H^0(\Gamma))$*.*

Remark 4.19. The previous proposition and Proposition 1.26, imply that the induction functor p_* is an extriangulated functor and that the equivalence in Proposition 4.18 induces an equivalence of 0-Auslander extriangulated categories. Indeed, recall than an extriangulated functor between two extriangulated categories $\mathcal{C}$ and $\mathcal{C}'$ is given by an additive functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\alpha : \mathbb{E}_{\mathcal{C}} \Rightarrow \mathbb{E}_{\mathcal{C}'} \circ (\mathcal{F}^{op} \times \mathcal{F})$ such that for any $X, Z \in \mathcal{C}$ and $\delta \in \mathbb{E}_{\mathcal{C}}(Z, X)$ realized by a conflation $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$, then $\alpha(\delta) \in \mathbb{E}_{\mathcal{C}'}(\mathcal{F}(Z), \mathcal{F}(X))$ is realized by the conflation $\mathcal{F}(X) \overset{\mathcal{F}(f)}{\rightarrowtail} \mathcal{F}(Y) \overset{\mathcal{F}(g)}{\twoheadrightarrow} \mathcal{F}(Z) \overset{\alpha(\delta)}{\dashrightarrow}$ ([BTS21, Defintion 2.32]). Moreover, an extriangulated functor $(\mathcal{F}, \alpha)$ is an extriangulated equivalence if and only if $\mathcal{F}$ is an additive equivalence and α is a natural isomorphism ([NOS22, Proposition 2.13]). In our case, the induction functor p_* is an additive equivalence between $\operatorname{per}^{[-1,0]}(\Gamma)$ and $\mathcal{K}^{[-1,0]}(\operatorname{proj} H^0(\Gamma))$ by Proposition 4.18. The corresponding natural transformation is the one making $p_* : \operatorname{per}(\Gamma) \to K^b(\operatorname{proj} H^0(\Gamma))$ into a triangulated (and hence extriangulated [BTS21, Theorem 2.33]) functor. By [FGP$^+$23, Theorem 2.8] for any $X, Z \in \operatorname{per}^{[-1,0]}(\Gamma)$ we have that

$$\mathbb{E}_{\operatorname{per}^{[-1,0]}(\Gamma)/\mathcal{I}}(Z, X) = \mathbb{E}_{\operatorname{per}^{[-1,0]}(\Gamma)}(Z, X),$$

and that conflations in $\operatorname{per}^{[-1,0]}(\Gamma)/\mathcal{I}$ are precisely the image of those in $\operatorname{per}^{[-1,0]}(\Gamma)$. Thus, the only thing left to verify is that for any $X, Z \in \operatorname{per}^{[-1,0]}(\Gamma)$ the natural transformation associated to p_* induces an isomorphism between

$$\operatorname{Hom}_{\operatorname{per}(\Gamma)}(Z, X[1]) \simeq \operatorname{Hom}_b(p_*(Z), p_*(X)[1]).$$

This follows essentially from [BY13, Proposition A.4]. We include a proof for the convenience of the reader.

Proposition 4.20. *[BY13, Proposition A.4] Let $X, Z \in \operatorname{per}^{[-1,0]}(\Gamma)$. Then the functor p_* induces an isomorphism*

$$\operatorname{Hom}_{\operatorname{per}(\Gamma)}(Z, X[1]) \simeq \operatorname{Hom}_b(p_*(Z), p_*(X)[1]).$$

Proof. Let $X, Z \in \operatorname{per}^{[-1,0]}(\Gamma)$. There are conflations

$$X' \rightarrowtail X \twoheadrightarrow X''[1] \tag{4.1.1}$$

$$Z' \rightarrowtail Z \twoheadrightarrow Z''[1] \tag{4.1.2}$$

with $X', X'', Z', Z'' \in \operatorname{add}_{\operatorname{per}(\Gamma)}(\Gamma)$. By applying the functor $\operatorname{Hom}_{\operatorname{per}(\Gamma)}(-, X''[1])$ and $\operatorname{Hom}_{K^b(\operatorname{proj} H^0(\Gamma))}(-, p_*(X'')[1])$ to the conflation 4.1.2 and its image under p_*, we get the following commutative diagram with exact rows

$$\begin{tikzcd}
\operatorname{Hom}(Z'[1], X''[1]) \arrow[r] \arrow[d, "f_1"] & \operatorname{Hom}(Z''[1], X''[1]) \arrow[r] \arrow[d, "f_2"] & \operatorname{Hom}(Z, X''[1]) \arrow[r] \arrow[d, "f_3"] & \operatorname{Hom}(Z', X''[1]) = 0 \arrow[d] \\
\operatorname{Hom}(p_*(Z')[1], p_*(X'')[1]) \arrow[r] & \operatorname{Hom}(p_*(Z'')[1], p_*(X'')[1]) \arrow[r] & \operatorname{Hom}(p_*(Z), p_*(X'')[1]) \arrow[r] & \operatorname{Hom}(p_*(Z'), p_*(X'')[1]) = 0.
\end{tikzcd}$$

Since p_* is an additive equivalence between $\operatorname{add}_{\operatorname{per}(\Gamma)}(\Gamma)$ and $\operatorname{add}_{\operatorname{per}(\bar{\Gamma})}(\bar{\Gamma})$, both f_1 and f_2 are isomorphisms, and by the Four Lemma then so if f_3. By applying $\operatorname{Hom}_{\operatorname{per}(\Gamma)}(-, X'[1])$ and $\operatorname{Hom}_{K^b(\operatorname{proj} H^0(\Gamma))}(-, p_*(X')[1])$ and using a similar argument, we get that p_* induces an isomorphism $\operatorname{Hom}(Z, X'[1]) \xrightarrow{g_3} \operatorname{Hom}(p_*(Z), p_*(X')[1])$. Now we apply $\operatorname{Hom}_{\operatorname{per}(\Gamma)}(Z, -)$ and $\operatorname{Hom}_{K^b(\operatorname{proj} H^0(\Gamma))}(p_*(Z), -)$ to the conflation 4.1.1 and its image under p_*, which produces de following commutative diagram with exact rows

$$\begin{tikzcd}
\operatorname{Hom}(Z, X''[1]) \arrow[r] \arrow[d, "f_3"] & \operatorname{Hom}(Z, X'[1]) \arrow[r] \arrow[d, "g_3"] & \operatorname{Hom}(Z, X[1]) \arrow[r] \arrow[d, "h"] & \operatorname{Hom}(Z, X''[2]) = 0 \arrow[d] \\
\operatorname{Hom}(p_*(Z), p_*(X'')[1]) \arrow[r] & \operatorname{Hom}(p_*(Z), p_*(X')[1]) \arrow[r] & \operatorname{Hom}(p_*(Z), p_*(X)[1]) \arrow[r] & \operatorname{Hom}(p_*(Z), p_*(X'')[2]) = 0.
\end{tikzcd}$$

Since f_3 and g_3 are isomorphisms, by the Four Lemma then so is h. ☐

The following corollary will be useful for the proof of Theorem 4.4.

Corollary 4.21. *Let $X, Z \in \operatorname{per}^{[-1,0]}(\Gamma)$ and suppose there is a conflation*

$$p_*(X) \rightarrowtail \bar{Y} \twoheadrightarrow p_*(Z) \overset{f}{\dashrightarrow} \tag{4.1.3}$$

where $\bar{Y} \in \mathcal{K}^{[-1,0]}(\operatorname{proj} H^0(\Gamma))$. Then there exists $Y \in \operatorname{per}^{[-1,0]}(\Gamma)$ and a conflation

$$X \rightarrowtail Y \twoheadrightarrow Z \overset{F}{\dashrightarrow}$$

whose image by p_* is the conflation 4.1.3. In particular, the image under p_* of any extension-closed subcategory $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj} \Lambda)$ is extension-closed.

Proof. Let $f \in \operatorname{Hom}_{\mathcal{K}^b(\operatorname{proj} H^0(\Gamma))}(p_*(Z), p_*(X)[1])$ and $p_*(X) \rightarrowtail \bar{Y} \twoheadrightarrow p_*(Z) \overset{f}{\dashrightarrow}$ be a conflation realizing f. By Proposition 4.20 there exist $F \in \operatorname{Hom}_{\operatorname{per}(\Gamma)}(Z, X[1])$ such that $p_*(F) = f$. By letting $Y = \operatorname{Cocone}(F)$ we get the result. □

Proposition 4.22. *[BY13, Proposition A.3] The induction functor* $p_* : \operatorname{per}(\Gamma) \to \mathcal{K}^b(\operatorname{proj} \bar{\Gamma})$ *induces a bijection between the sets of isomorphism classes of* 2*-term silting objects* 2-silt Γ *and* 2-silt $\bar{\Gamma}$.

4.1.4 Semi-bricks

In [Asa20], S. Asai studied how semi-bricks relate to the correspondences evoked in Chapter 3 and 2-term simple-minded collections. In this short section, we recall his results, which will be employed to show that the maps $\mathscr{T}$ and $\mathscr{W}$ from Proposition 3.70 are inverse of each other when restricted to the sets of left finite wide subcategories and thick subcategories with enough injectives.

Definition 4.23. Let Λ be a finite-dimensional $\Bbbk$-algebra. A module $S \in \operatorname{mod} \Lambda$ is called a *brick* if $\operatorname{End}_\Lambda(S)$ is a division algebra. We write brick Λ for the set of isomorphism classes of bricks in $\operatorname{mod} \Lambda$. A subset $\mathcal{S} \subset$ brick Λ is called a *semibrick* if $\operatorname{Hom}_\Lambda(S, S') = 0$ for any $S \neq S'$ in $\mathcal{S}$. We denote by sbrick Λ the set of semibricks in $\operatorname{mod} \Lambda$.

Examples of semibricks include the set of simple modules $\{S_1, \cdots, S_n\}$ in $\operatorname{mod} \Lambda$. In general, for any wide subcategory $\mathcal{W}$ of $\operatorname{mod} \Lambda$, the set of simple objects of $\mathcal{W}$ is a semibrick. This assignation gives rise to a bijection between the sets sbrick Λ and wide Λ whose inverse sends $\mathcal{S} \in$ sbrick Λ to $\operatorname{Filt}(\mathcal{S})$ [Asa20, Proposition 1.24]. We say that a semibrick $\mathcal{S}$ is *left finite* if $\operatorname{Filt}(\operatorname{Fac}(\mathcal{S}))$ is a functorially finite torsion class. We denote by l-sbrick Λ the set of left finite semibricks in $\operatorname{mod} \Lambda$.

For any basic τ-rigid module $M \in \operatorname{mod} \Lambda$ let $\mathcal{S}(M)$ be the set of indecomposable summands of $M / \operatorname{rad}_{\operatorname{End}_\Lambda(M)}(M)$, we have the following result.

Theorem 4.24. *[Asa20, Theorem 1.3] Let* Λ *be a finite-dimensional algebra. Then the map*

$$\begin{aligned} \text{s}\tau\text{-tilt}\,\Lambda &\to \text{l-sbrick}\,\Lambda \\ (M, P) &\mapsto \mathcal{S}(M) \end{aligned}$$

is a bijection which is compatible with the correspondences in Theorem 3.1. That is, there is a commutative diagram

$$\begin{tikzcd} \text{s}\tau\text{-tilt}\,\Lambda \arrow[r, "\text{Fac}"] \arrow[d, "\mathcal{S}"] & \text{f-tors}\,\Lambda \arrow[d, "\alpha"] \\ \text{l-sbrick}\,\Lambda \arrow[r, "\text{Filt}"] & \text{l-wide}\,\Lambda. \end{tikzcd}$$

.

The following is an excerpt of [Asa20, Theorem 2.3] which, among other things, establishes the relationship between left finite semibricks, 2-term silting objects and 2-term simple-minded collections.

Theorem 4.25. *[Asa20, Theorem 2.3] Let Λ be a finite-dimensional algebra. Then the map that takes a* 2*-term simple-minded collection* $\mathcal{S} \in 2\text{-smc}\,\Lambda$ *and sends it to the set* $\mathcal{S} \cap \operatorname{mod}\Lambda$ *induces a bijection between the sets* $2\text{-smc}\,\Lambda$ *and* $\text{l-sbrick}\,\Lambda$*. Moreover, this bijection fits in the following commutative diagram*

$$\begin{tikzcd}
2\text{-silt}\,\Lambda \arrow[r, "\Omega"] \arrow[d, "H^0"] & 2\text{-smc}\,\Lambda \arrow[d, "-\cap \operatorname{mod}\Lambda"] \\
\text{s}\tau\text{-tilt}\,\Lambda \arrow[r, "\mathcal{S}"] \arrow[d, "\operatorname{Fac}"] & \text{l-sbrick}\,\Lambda \arrow[d, "\operatorname{Filt}"] \\
\text{f-tors}\,\Lambda \arrow[r, "\alpha"] & \text{l-wide}\,\Lambda.
\end{tikzcd}$$

4.1.5 τ^{-1}-tilting theory

Recall that a module $M \in \operatorname{mod}\Lambda$ is said to be τ^{-1}-rigid if $\operatorname{Hom}_\Lambda(\tau^{-1}M, M) = 0$. We say that a pair (M, I) where $M \in \operatorname{mod}\Lambda$ and $I \in \operatorname{inj}\Lambda$ is τ^{-1}-rigid, if M is τ^{-1}-rigid and $\operatorname{Hom}_\Lambda(M, I) = 0$. We say that (M, I) is support τ^{-1}-tilting if moreover $|M| + |I| = |\Lambda|$. We denote by $\text{s}\tau^{-1}\text{-tilt}\,\Lambda$ the set of isoclasses of support τ^{-1}-rigid pairs. A support τ^{-1}-rigid pair (M, I) gives rise to a torsion pair given by

$$({}^{\perp}M, \operatorname{Sub}(M)),$$

where $\operatorname{Sub}(M) = \{N \in \operatorname{mod}\Lambda \mid N \subset M^n \text{ for some } n \in \mathbb{N}\}$ is a functorially finite torsion-free class. We denote by $\text{f-torsf}\,\Lambda$ the set of functorially finite torsion free classes in $\operatorname{mod}\Lambda$.

Theorem 4.26. *[AIR14, Theorem 2.15] Let Λ be a finite-dimensional algebra. Then we have bijections*

$$\begin{aligned}
\operatorname{Sub} : \text{s}\tau^{-1}\text{-tilt}\,\Lambda &\longrightarrow \text{f-torsf}\,\Lambda \\
(M, I) &\longmapsto \operatorname{Sub}(M) \\
\Delta : \text{s}\tau\text{-tilt}\,\Lambda &\longrightarrow \text{s}\tau^{-1}\text{-tilt}\,\Lambda \\
(M, P) &\longmapsto (\tau M \oplus \nu P, I)
\end{aligned}$$

where I is the largest basic injective object such that $\operatorname{Hom}_\Lambda(\tau M \oplus \nu P, I) = 0$*. Moreover, these bijections satisfy that for all* $(M, P) \in \text{s}\tau\text{-tilt}\,\Lambda$

$$\operatorname{Fac}(M) = {}^{\perp}(\tau M \oplus \nu P).$$

In particular, the torsion classes $(\operatorname{Fac}(M), M^{\perp})$ *and* $({}^{\perp}(\tau M \oplus \nu P), \operatorname{Sub}(\tau M \oplus \nu P))$ *coincide.*

We say that a wide subcategory $\mathcal{W}$ (respectively a semibrick $\mathcal{S}$) is *right finite* if $\operatorname{Filt}(\operatorname{Sub}(\mathcal{W}))$ (respectively $\operatorname{Filt}(\operatorname{Sub}(\mathcal{S}))$) is a functorially finite torsion-free class. We denote by $\text{r-wide}\,\Lambda$ and $\text{r-sbrick}\,\Lambda$ the set of right finite wide subcategories and semibricks respectively.

Theorem 4.27. *[Asa20, Theorem 2.3] Let Λ be a finite-dimensional algebra. Then the map that takes a 2-term simple-minded collection $\mathcal{S} \in$ 2-smc Λ and sends it to the set $\mathcal{S}[-1] \cap \operatorname{mod}\Lambda$ induces a bijection between the sets 2-smc Λ and r-sbrick Λ. Moreover, this bijection fits in the following commutative diagram*

$$\begin{tikzcd}
\text{2-silt}\,\Lambda \arrow[r, "\Omega"] \arrow[d, "H^{-1}(\nu-)"] & \text{2-smc}\,\Lambda \arrow[d, "-[-1]\cap\operatorname{mod}\Lambda"] \\
\text{s}\tau^{-1}\text{-tilt}\,\Lambda \arrow[r, "\mathcal{S}'"] \arrow[d, "\text{Sub}"] & \text{r-sbrick}\,\Lambda \arrow[d, "\text{Filt}"] \\
\text{f-torsf}\,\Lambda & \text{r-wide}\,\Lambda \arrow[l, "\text{Filt}(\text{Sub}(-))"]
\end{tikzcd}$$

where $\mathcal{S}'$ takes (M, I) to the set of indecomposable summands of $\operatorname{soc}_{\operatorname{End}_\Lambda(M)}(M)$.

4.2 Silting reduction in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$

The goal of this section is to explicitly describe the reduction of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with respect to a presilting object U. Reductions for hereditary extriangulated categories were treated in general in [GNP23, Section 2.2.3]. In our setting, we show that the reduction of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ by a 2-term presilting complex is equivalent to the 2-term category of perfect complexes over a non-positive dg algebra. This is done using Iyama-Yoshino reduction in $\mathcal{K}^b(\operatorname{proj}\Lambda)$ [IY08] as well as Iyama-Yang's results showing that the Verdier localisation by a presilting object is a reduction [IY18]. We show that both operations are compatible with those in [GNP23]. This is a particular case of the reduction of a 0-Auslander triangulated category $\mathcal{K}$ with respect to a presilting object U being equivalent to the Verdier localization $\mathcal{K}/\operatorname{thick}(U)$, which was shown in general in [Bør24].

4.2.1 τ-tilting reduction

In Chapter 3 we recalled how a support τ-tilting pair (M, P) gave rise to a wide subcategory, namely $\mathscr{W}_{(M_\lambda, P)} = {}^{\perp}\tau M_\lambda \cap P^{\perp} \cap M_\lambda^{\perp}$, where M_λ is a direct summand of M. The subsequent definition extends this construction to any support τ-rigid pair.

Definition 4.28. Let (M, P) be a support τ-rigid pair. The *τ-tilting reduction* of $\operatorname{mod}\Lambda$ with respect to (M, P) is the subcategory of $\operatorname{mod}\Lambda$ given by

$$\mathscr{W}_{(M,P)} = {}^{\perp}\tau M \cap P^{\perp} \cap M^{\perp}.$$

The concept of τ-tilting reduction was introduced and thoroughly studied by G. Jasso in [Jas15]. This brief section serves to review some of its key properties.

Proposition 4.29. *[DIR$^+$23, Theorem 3.28] For any support τ-rigid pair (M, P), the category $\mathscr{W}_{(M,P)}$ is wide.*

Proof. Let $X_M \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be the minimal projective presentation of M. By Proposition 3.24, we know that $\mathscr{W}_{(M,P)} = \mathscr{W}(X_M \oplus P[1])$ which is a wide subcategory by Proposition 3.70. □

Recall that any 2-term presilting complex can be completed into a silting complex in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ [Aih13, Proposition 2.16]. Let $f : U' \to \Lambda[1]$ be a minimal right $\operatorname{add}(U)$-approximation and let $V' = \operatorname{Cocone}(f) \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then $U \oplus V$ is a silting object, where V is the basic module satisfying $\operatorname{add}(V) = \operatorname{add}(V')$. We call $T_U = U \oplus V$ the *Bongart'z completion* of U and we refer to V as the Bongartz complement of U (see Lemma 3.65). If $M = H^0(U)$ and $P \in \operatorname{proj}\Lambda$ is the basic object satisfying $\operatorname{add}(P) = \operatorname{add}(U[-1]) \cap \operatorname{add}(\Lambda)$, then $N = H^0(V)$ is the Borngartz's complement of the support τ-rigid pair (M, P), that is $(M \oplus N, P)$ is support τ-tilting and $T_U = P({}^{\perp}(\tau U) \cap P^{\perp})$ [DIR+23, Theorem 4.4].

Theorem 4.30. *[DIR+23, Theorem 4.12] [Jas15, Theorem 3.8] Let (M, P) is a support τ-rigid pair in $\operatorname{mod}\Lambda$ and let $C_{(M,P)} = \operatorname{End}_\Lambda(M \oplus N)/e_M$ where N is the Bongartz complement of (M, P) and e_M the idempotent corresponding to $\operatorname{Hom}_\Lambda(M \oplus N, M)$. Then the map*

$$\operatorname{Hom}_\Lambda(M \oplus N, -) : \operatorname{mod}\Lambda \to \operatorname{mod} C_{(M,P)}$$

induces an equivalence

$$\mathscr{W}_{(M,P)} \xrightarrow{\simeq} \operatorname{mod} C_{(M,P)}.$$

Theorem 4.31. *[BM21, Theorem 3.6][Jas15, Theorem 3.16] Let (M, P) be a support τ-rigid pair in $\operatorname{mod}\Lambda$ and let $C_{(M,P)}$ be as in Theorem 4.30. Then there are bijections*

$$\begin{tikzcd}\{(N,Q) \in \mathrm{s}\tau\text{-rigid}\,\Lambda \mid (N \oplus M, Q \oplus P) \in \mathrm{s}\tau\text{-rigid}\,\Lambda \text{ and } |N| + |Q| = 1\} \arrow[d, "\mathcal{E}_{(M,P)}"] \\ \{(L,R) \in \mathrm{s}\tau\text{-rigid}\, C_{(M,P)} \mid |L| + |R| = 1\}\ .\end{tikzcd}$$

In particular, if Λ is g-finite, then so is $C_{(M,P)}$.

4.2.2 Thick subcategories generated by 2-term presilting complex

Recall that a subcategory $\mathcal{T}$ of a triangulated category $\mathcal{D}$ with shift functor Σ is *thick* if it is full, closed under direct summands and if it inherits a tringulated structure from $\mathcal{D}$. For any full subcategory $\mathcal{C}$ of $\mathcal{D}$, we denote by $\operatorname{thick}_{\mathcal{D}}(\mathcal{C})$ the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{C}$. Let $\mathcal{K} \subset \mathcal{D}$ be full and closed under extensions, and thus, extriangulated [NP19]. The next proposition follows from the definitions.

Proposition 4.32. *Let $\mathcal{T} \subset \mathcal{D}$ be a thick subcategory of $\mathcal{D}$ and let $\mathcal{K} \subset \mathcal{D}$ be closed under extensions and direct summands. Then $\mathcal{T} \cap \mathcal{K}$ is a thick subcategory of the extriangulated category $\mathcal{K}$.*

Proposition 4.33. *Let $\mathcal{D}$ be a Hom-finite, Krull-Schmidt triangulated category. If $\mathcal{U} = \operatorname{add}(\mathcal{U})$ is a presilting subcategory of $\mathcal{C}$, then*

(i) [IY18, Propositions 2.7] For all $n \leq 0$,

$$\mathcal{U} * \mathcal{U}[1] * \cdots * \mathcal{U}[n] = \operatorname{add}(\mathcal{U} * \mathcal{U}[1] * \cdots * \mathcal{U}[n]).$$

(ii) [AI12, Propositions 2.15]

$$\begin{aligned}\operatorname{thick}_{\mathcal{D}}(\mathcal{U}) &= \bigcup_{n\geq 0} \operatorname{add}\left(\mathcal{U}[-n] * \mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n]\right)\\ &= \bigcup_{n\geq 0} \mathcal{U}[-n] * \mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n].\end{aligned}$$

In what follows, $\mathcal{D} = \mathcal{K}^b(\operatorname{proj}\Lambda)$ and $\mathcal{K} = \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Since we will work with the notion of thick subcategory in the triangulated category $\mathcal{K}^b(\operatorname{proj}\Lambda)$ as well as the notion of thick subcategory in the extriangulated category $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, to avoid confusion, for any subcategories $\mathcal{C} \subset \mathcal{K}^b(\operatorname{proj}\Lambda)$ and $\mathcal{T} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ we will denote $\operatorname{thick}_b(\mathcal{C})$ the smallest (triangulated) thick subcategory in $\mathcal{K}^b(\operatorname{proj}\Lambda)$ containing $\mathcal{C}$, and by $\operatorname{thick}_{[-1,0]}(\mathcal{T})$ the smallest (extringulated) thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ that contains $\mathcal{T}$.

Lemma 4.34. *Let $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a 2-term presilting complex and $\mathcal{U} = \operatorname{add}(U)$, then*

$$\operatorname{thick}_{[-1,0]}(U) = \operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = (\mathcal{U}[-1] * \mathcal{U} * \mathcal{U}[1]) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda).$$

Proof. It follows from the definitions that $\operatorname{thick}_{[-1,0]}(U) \subset \operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Let $X \in (\mathcal{U}[-1] * \mathcal{U} * \mathcal{U}[1]) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, then there exists a triangle

$$V[-1] \to X \to Y \dashrightarrow V \tag{4.2.1}$$

where $V \in \mathcal{U}$ and $Y \in \mathcal{U} * \mathcal{U}[1]$. In particular Y is a 2-term complex in $\mathcal{U} * \mathcal{U}[1]$. By definition, there exists a triangle

$$\bar{U}' \to Y \to \bar{U}[1] \dashrightarrow \bar{U}'[1] \tag{4.2.2}$$

with $\bar{U}, \bar{U}' \in \mathcal{U}$. A rotation of the previous triangle gives a conflation $\bar{U} \rightarrowtail \bar{U}' \twoheadrightarrow Y$ in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, which implies that $Y \in \operatorname{thick}_{[-1,0]}(U)$. By rotating the triangle 4.2.1, we get $X \rightarrowtail Y \twoheadrightarrow V$, implying that $X \in \operatorname{thick}_{[-1,0]}(U)$. Thus

$$(\mathcal{U}[-1] * \mathcal{U} * \mathcal{U}[1]) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \subset \operatorname{thick}_{[-1,0]}(U).$$

We show now that $\operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \subset \mathcal{U}[-1] * \mathcal{U} * \mathcal{U}[1]$. Since $\mathcal{U}$ is presilting, by Proposition 4.33 we have that

$$\begin{aligned}\operatorname{thick}_b(U) = \operatorname{thick}_b(\mathcal{U}) &=\\ &= \bigcup_{n\geq 0} \mathcal{U}[-n] * \mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n].\end{aligned}$$

Let $X \in \operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then there exists n such that $X \in \mathcal{U}[-n] * \mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n]$. Since taking extensions is an associative operation, we can find a triangle

$$U'[-n] \xrightarrow{f} X \to X' \dashrightarrow U'[1-n]$$

where $U' \in \mathcal{U}$ and $X' \in \mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n]$. Suppose that $n > 1$. Since both X and U are 2-term complexes, $X \in {}^{\perp}\mathcal{U}[\leq 2] \cap \mathcal{U}[\geq 2]^{\perp}$. This implies that

$f = 0$ and thus, X is a direct summand of X'. By Proposition 4.33 i), we know that $\mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n] = \operatorname{add}(\mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n])$ and thus $X \in \mathcal{U}[1-n] * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n]$. By applying the previous argument whenever $i - n < -1$, we can deduce that $X \in \mathcal{U}[-1] * \mathcal{U} * \cdots * \mathcal{U}[n-1] * \mathcal{U}[n]$. Using again the associativity of taking extension, we can find a triangle

$$X'' \to X \xrightarrow{g} U''[n] \dashrightarrow X''[1].$$

where $X'' \in \mathcal{U}[-1] * \mathcal{U} * \cdots * \mathcal{U}[n-1]$ and $U'' \in \mathcal{U}$. Once more, if $n > 1$ we deduce that $g = 0$, since both X and U'' are 2-term complexes, and thus X is a direct summand of $X'' \in \mathcal{U}[-1] * \mathcal{U} * \cdots * \mathcal{U}[n-1] = \operatorname{add}(\mathcal{U}[-1] * \mathcal{U} * \cdots * \mathcal{U}[n-1])$. We conclude that $X \in \mathcal{U}[-1] * \mathcal{U} * \cdots * \mathcal{U}[n-1]$. Applying this argument recursively whenever $n - i > 1$, we finally get that $X \in \mathcal{U}[-1] * \mathcal{U} * \mathcal{U}[1]$. We conclude that

$$\begin{aligned}(\mathcal{U}[-1] * \mathcal{U} * \mathcal{U}[1]) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) &\subset \operatorname{thick}_{[-1,0]}(U) \\ &\subset \operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \subset (\mathcal{U}[-1] * \mathcal{U} * \mathcal{U}[1]) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda),\end{aligned}$$

which gives the result.

□

Corollary 4.35. *Let $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a thick subcategory and consider $U \in \mathcal{H}$ a 2-term presilting object. If $\operatorname{thick}_b(U) = \operatorname{thick}_b(\mathcal{H})$, then $\operatorname{thick}_{[-1,0]}(U) = \mathcal{H}$.*

Proof. Suppose $\operatorname{thick}_b(U) = \operatorname{thick}_b(\mathcal{H})$ and let $\mathcal{U} = \operatorname{add}(U)$. Since $U \in \mathcal{H}$ and since $\mathcal{H}$ is thick, we have that $\operatorname{thick}_{[-1,0]}(U) \subset \mathcal{H}$. For the other inclusion note that $\mathcal{H} \subset \operatorname{thick}_b(\mathcal{H}) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. By Lemma 4.34, $\operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \operatorname{thick}_{[-1,0]}(U)$, which gives the result. □

4.2.3 Silting reduction in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$

Before introducing the notion of reduction in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, let us first recall O. Iyama and D. Yang's additive description of the reduction of a triangulated category with respect to a presilting subcategory.

Theorem 4.36. *[IY18, Theorem 1.1] Let $\mathcal{D}$ be a triangulated category and let $\mathcal{U}$ be a presilting subcategory of $\mathcal{D}$ satisfying certain mild assumptions*[2]*. Let $\mathcal{J}_{\mathcal{U}} = \mathcal{D}/\operatorname{thick}_{\mathcal{D}}(\mathcal{U})$ the triangle quotient of $\mathcal{D}$ with respect to $\mathcal{U}$. Then the additive quotient $\mathcal{Z}_U/[\mathcal{U}]$ for $\mathcal{Z}_{\mathcal{U}} = ({}^{\perp_{\mathcal{D}}}\mathcal{U}[>0]) \cap (\mathcal{U}[<0]^{\perp_{\mathcal{D}}})$ has a natural structure of a triangulated category and we have a triangle equivalence $\mathcal{Z}_{\mathcal{U}}/[\mathcal{U}] \xrightarrow{\bar{\rho}} \mathcal{J}_{\mathcal{U}}$, where $\bar{\rho}$ is induced by the localisation functor $\mathcal{Z}_{\mathcal{U}} \subset \mathcal{D} \xrightarrow{\rho} \mathcal{D}/\operatorname{thick}_{\mathcal{D}}(\mathcal{U}) = \mathcal{J}_{\mathcal{U}}$.*

Theorem 4.37. *[IY18, Theorem 3.7] Under the assumptions of Theorem 4.36, the localisation functor $\rho : \mathcal{D} \to \mathcal{J}_{\mathcal{U}}$ induces a bijection between the sets of presilting subcategories in $\mathcal{J}_U$ and presilting subcategories in $\mathcal{D}$ containing $\mathcal{U}$. Moreover, a subcategory $\mathcal{P} \subset \mathcal{D}$ containing $\mathcal{U}$ is silting if and only if $\rho(\mathcal{P})$ is silting in $\mathcal{J}_{\mathcal{U}}$.*

---

[2] These assumptions are satisfied if, for instance, $\mathcal{D}$ is Hom-finite over a field and $\mathcal{U} = \operatorname{add}(U)$ for certain $U \in \mathcal{D}$ that can be completed into a silting object. Since we are working in the context where $\mathcal{K}$ is a 0-Auslander reduced extriangulated category [GNP23], and thus equivalent to $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ for certain finite-dimensional algebra Λ [Che23], the needed assumptions hold (for more on these hypotheses, see [IY18, Section 3.1]).

From now on we suppose that $\mathcal{D} = \mathcal{K}^b(\operatorname{proj}\Lambda)$ and $\mathcal{U} = \operatorname{add}(U)$ where U is a 2-term presilting complex. Under these hypothesis, the category $\mathcal{J}_\mathcal{U}$ has the following explicit description.

Proposition 4.38. *[Bør21, Proposition 2.11] Let Λ be a finite-dimensional algebra and let U be a basic 2-term presilting complex in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ with Bongartz completion T_U. Then $\mathcal{J}_\mathcal{U} = \mathcal{K}^b(\operatorname{proj}\Lambda)/\operatorname{thick}_b(\mathcal{U})$ is equivalent to the category of perfect complexes $\operatorname{per}(C_U)$ where C_U is the dg algebra quotient $\mathcal{E}nd_\mathcal{D}(T_U)/\langle e_U\rangle$.*

Remark 4.39. The choice of C_U as notation is intentional. Let (M,P) be the support τ-rigid pair associated to U, then by [Jas15, Theorem 4.12 b)]

$$H^0(C_U) = \operatorname{End}_{\mathcal{J}_\mathcal{U}}(T_U) \simeq \operatorname{End}_\Lambda(H^0(T_U))/\langle e_{H^0(U)}\rangle,$$

where $\operatorname{End}_\Lambda(H^0(T_U))/\langle e_{H^0(U)}\rangle = C_{(M,P)}$ is the algebra associated to the τ-tilting reduction of $\operatorname{mod}\Lambda$ by (M,P) as seen in Theorem 4.30.

The following proposition is a weaker version of [IY18, Lemma 3.4] which will be essential four our results.

Proposition 4.40. *[IY18, Lemma 3.4] The localisation functor $\mathcal{K}^b(\operatorname{proj}\Lambda) \xrightarrow{\rho} \mathcal{J}_\mathcal{U}$ induces a bijective map*

$$\operatorname{Hom}_\mathcal{D}(X, Y[i]) \to \operatorname{Hom}_{\mathcal{J}_\mathcal{U}}(X, Y[i])$$

for every $i > 0$ and $X, Y \in \mathcal{Z}_\mathcal{U}$.

Moreover, it is the case that any triangle $X \xrightarrow{\bar{f}} Y \xrightarrow{\bar{g}} Z \dashrightarrow X\langle 1\rangle$ in $\mathcal{J}_\mathcal{U}$ is the image of a triangle in $\mathcal{Z}_U/[\mathcal{U}]$, which is in turn isomorphic to triangles obtained from a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, dashed] \arrow[d] & X[1] \arrow[d, equal] \\
X \arrow[r, "p_U"] & U_X \arrow[r] & X\langle 1\rangle \arrow[r, dashed] & X[1]\,,
\end{tikzcd}$$

where p_U is a minimal left $\mathcal{U}$-approximation of X. This fact says in particular that if $\mathcal{C} \subset \mathcal{Z}_U$ is thick in $\mathcal{Z}_\mathcal{U}$, this remains true for $\mathcal{C}/[U]$ in $\mathcal{Z}_\mathcal{U}/[U]$ and thus for $\rho(\mathcal{C}) \subset \rho(\mathcal{Z}_\mathcal{U})$.

The following lemma was originally shown by O. Iyama and D. Yang as a step towards proving Theorem 4.36. In their context, they establish that for any $X \in \mathcal{D}$ there exists $Y \in \mathcal{Z}_\mathcal{U}$ such that $X \simeq Y$ in $\mathcal{J}_\mathcal{U}$. In the following lemma, we adapt their arguments to show that if $\mathcal{D} = \mathcal{K}^b(\operatorname{proj}\Lambda)$, and $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, then Y can be chosen from $\mathcal{Z}_\mathcal{U} \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.

Lemma 4.41. *[IY18, Lemma 3.3] Let Λ be a finite-dimensional algebra and let U be a basic 2-term presilting complex in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. For any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ there exists $Y \in \mathcal{Z}_\mathcal{U}^{[-1,0]} := \mathcal{Z}_\mathcal{U} \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfying $X \simeq Y$ in $\mathcal{J}_\mathcal{U}$.*

Proof. Let $\mathcal{U} = \operatorname{add}(U)$, with U basic and presilting. First note that

$$\mathcal{Z}_\mathcal{U} = \{X \in \mathcal{D} \mid \operatorname{Hom}_\mathcal{D}(X, U[i]) = 0 = \operatorname{Hom}_\mathcal{D}(U[-i], X)\ \forall i > 0\}.$$

Since $\operatorname{Hom}_{\mathcal{D}}(Y, Y[i]) = \operatorname{Hom}_{\mathcal{D}}(Y[-i], Y) = 0$ for all $i \geq 2$ and all $Y \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, then

$$\mathcal{Z}_{\mathcal{U}}^{[-1,0]} = \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid \mathbb{E}(X, U) = 0 = \mathbb{E}(U, X)\}.$$

Recall that $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) = \Lambda * \Lambda[1]$ and that Λ and T_U are isomorphic in $\mathcal{J}_{\mathcal{U}}$ where T_U is the Bongartz completion of U into a silting complex in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, which by definition lies in $\mathcal{Z}_{\mathcal{U}}^{[-1,0]}$. We will show that we can find $H \in \mathcal{Z}_{\mathcal{U}}^{[-1,0]}$ such that $T_U[1] \simeq H$ in $\mathcal{J}_{\mathcal{U}}$. Consider the conflation $\Lambda \rightarrowtail U' \twoheadrightarrow V \dashrightarrow \Lambda[1]$ where the first morphism is a minimal right $\mathcal{U}$-approximation of Λ, then by definition $V \in \mathcal{K}$ and $V \simeq \Lambda[1] \simeq T_U[1]$ in $\mathcal{J}$. In fact, $V \simeq T_U^c$ inside $\mathcal{J}_{\mathcal{U}}$, where T_U^c is the Bongartz co-completion of U, that is, the basic silting complex satisfying that $\operatorname{add}(T_U^c) = \operatorname{add}(V \oplus U)$. Since $T_U^c \in \mathcal{K}$ and $\mathbb{E}(T_U^c, U) = 0 = \mathbb{E}(U, T_U^c)$, we have that $T_U^c \in \mathcal{Z}_{\mathcal{U}}^{[-1,0]}$ and that $T_U * T_U^c \subset \mathcal{Z}_{\mathcal{U}}^{[-1,0]}$. Since the functor $\bar{\rho}$ is a triangle equivalence, by Proposition 4.40 we have that

$$\mathcal{Z}_{\mathcal{U}}^{[-1,0]}/[\mathcal{U}] \supseteq (T_U * T_U^c)/[\mathcal{U}] \xrightarrow{\bar{\rho}} \rho(\Lambda) *_{\mathcal{J}} \rho(\Lambda)\langle 1\rangle \simeq \rho(\Lambda * \Lambda[1]) = \rho(\mathcal{K}) \subseteq \mathcal{J}.$$

In particular, for each $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ there exists $X' \in T_U * T_U^c$ such that $\rho(X) \simeq \rho(X')$. ☐

Remark 4.42. We note that the full extension-closed subcategory $\mathcal{Z}_{\mathcal{U}}^{[-1,0]}$ is precisely the subcategory considered by M. Gorsky, H. Nakaoka and Y. Palu in [GNP23, Definition 2.7] to define the reduction of an extriangulated category with respect to a presilting object.

Theorem 4.43. *Let $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a 2-term presilting complex and $\mathcal{U}$ its additive closure. Consider $\mathcal{Z}_{\mathcal{U}}$ and $\rho : \mathcal{K}^b(\operatorname{proj}\Lambda) \to \mathcal{K}^b(\operatorname{proj}\Lambda)/\operatorname{thick}_b(\mathcal{U}) \simeq \operatorname{per}(C_U)$ as in Theorem 4.36 and Proposition 4.38. Let $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a thick subcategory such that $U \in \mathcal{H}$. Then*

$$\frac{\mathcal{H} \cap \mathcal{Z}_{\mathcal{U}}}{[U]} \simeq \rho(\mathcal{H}).$$

In particular, $\rho(\mathcal{H})$ is thick inside the extriangulated category

$$\rho(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) \simeq \operatorname{per}^{[-1,0]}(C_U).$$

Proof. The following argument follows closely those in [IY18, Proposition 3.2, Lemma 3.3]. We are going to show that for every $X \in \mathcal{H}$ there exists an object $Z \in \mathcal{H} \cap \mathcal{Z}_{\mathcal{U}} \subset \mathcal{Z}_{\mathcal{U}}^{[-1,0]}$ such that $\rho(X) \simeq \rho(Z)$. Let $X \in \mathcal{H}$ and take $\mathcal{X} \xrightarrow{f} U'[1]$ a minimal left $\mathcal{U}[1]$-approximation and let $Y = \operatorname{Cocone}(f)$. We then can construct a conflation

$$U' \rightarrowtail Y \twoheadrightarrow X \overset{f}{\dashrightarrow},$$

which implies that $Y \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $\operatorname{Hom}_b(Y, U[n]) = 0$ for all $n \geq 2$. Moreover, applying the functor $\operatorname{Hom}_{\mathcal{D}}(-, U[1])$ to the previous triangle we obtain an exact sequence

$$\operatorname{Hom}_{\mathcal{D}}(U'[1], U[1]) \xrightarrow{\operatorname{Hom}_{\mathcal{D}}(-,f)} \operatorname{Hom}_{\mathcal{D}}(X, U[1]) \to \operatorname{Hom}_{\mathcal{D}}(Y, U[1]) \to \operatorname{Hom}_{\mathcal{D}}(U', U[1]).$$

Given that $\operatorname{Hom}_{\mathcal{D}}(U', U[1]) = 0$, since U is presilting and since $\operatorname{Hom}_{\mathcal{D}}(-, f)$ is surjective because f is a left $\mathcal{U}[1]$-approximation, we deduce that $\operatorname{Hom}_{\mathcal{D}}(Y, U[1]) = 0$, Now, consider $U''[-1] \xrightarrow{g} Y$ a minimal right $\mathcal{U}[-1]$-approximation and let $Z = \operatorname{Cone}(g)$, then there exists a conflation

$$Y \rightarrowtail Z \twoheadrightarrow U'' \overset{g[1]}{\dashrightarrow} .$$

In particular, $Z \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ and $Z \in U[< -1]^{\perp}$. As before, by applying the functor $\operatorname{Hom}_{\mathcal{D}}(U[-1], -)$, and using that g is a right $\mathcal{U}[-1]$-approximation and U silting, we can deduce that $Z \in U[-1]^{\perp}$. But Z is an extension between two objects in ${}^{\perp}U[1]$, which implies that $Z \in {}^{\perp}\mathcal{U}[> 0] \cap \mathcal{U}[< 0]^{\perp} = \mathcal{Z}_{\mathcal{U}}$. Moreover, under the assumption that $U \in \mathcal{H}$, both previous conflations give that both Y and Z lie in $\mathcal{H}$. We get that

$$\rho(X) \simeq \rho(Y) \simeq \rho(Z),$$

with $Z \in \mathcal{H} \cap \mathcal{Z}_{\mathcal{U}}$, and hence $\frac{\mathcal{H} \cap \mathcal{Z}_{\mathcal{U}}}{[U]} \simeq \rho(\mathcal{H})$. Since $\mathcal{H} \cap \mathcal{Z}_{\mathcal{U}}$ is thick in $\mathcal{Z}_{\mathcal{U}}^{[-1,0]}$ we get that

$$\rho(\mathcal{H}) \simeq \frac{\mathcal{H} \cap \mathcal{Z}_{\mathcal{U}}}{[U]} \subset \frac{\mathcal{Z}_{\mathcal{U}}^{[-1,0]}}{[U]} \simeq \rho(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$$

is thick in $\rho(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)) \simeq \operatorname{per}^{[-1,0]}(C_U)$. □

4.3 Thick subcategories and cotorsion pairs of g-finite algebras

In this section we assume that Λ is a g-finite algebra and study the maps between cotorsion pairs, thick subcategories and presilting objects in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ introduced in [Gar23].

4.3.1 Bijection between cotorsion and torsion classes

The following result shows that the map between torsion pairs and cotorsion pairs introduced by D. Pauksztello and A. Zvonareva and reviewed in Chapter 3 is a bijection.

Theorem 4.44. *Let Λ be a finite-dimensional $\Bbbk$-algebra and consider $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then the functor $H^0 : \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \to \Lambda$ induces a bijection*

$$\begin{aligned} H^0 : \operatorname{cotor}\Lambda &\to \operatorname{tors}\Lambda \\ (\mathcal{X}, \mathcal{Y}) &\mapsto H^0(\mathcal{Y}). \end{aligned}$$

Proof. In [PZ23], the authors show that this map is well defined and that it induces a bijection between the set of complete cotorsion pairs and that of functorially finite torsion classes. We are going to show that it is in fact always a bijection whose inverse map is given by

$$\mathcal{T} \mapsto \left({}^{\perp_1}\mathcal{Y}_{\mathcal{T}}, \mathcal{Y}_{\mathcal{T}}\right)$$

where $\mathcal{Y}_{\mathcal{T}} = \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid H^0(X) \in \mathcal{T}\}$. It suffices to show that $({}^{\perp_1}\mathcal{Y}_{\mathcal{T}})^{\perp_1} = \mathcal{Y}_{\mathcal{T}}$. Remark that for any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, $X \simeq X_M \oplus Q[1]$, where X_M is the minimal projective presentation of $M = H^0(X)$ and $Q \in \operatorname{proj}\Lambda$. Then by Proposition 3.29 any $X, Y \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ satisfy

$$\begin{aligned} \mathbb{E}(X, Y) &\simeq \mathbb{E}(X_M, Y) \oplus \mathbb{E}(Q[1], Y) \simeq D\operatorname{Hom}_{\Lambda}(H^0(Y), \tau M) \oplus \operatorname{Hom}_b(Q, Y) \\ &\simeq D\operatorname{Hom}_{\Lambda}(H^0(Y), \tau M) \oplus \operatorname{Hom}_{\Lambda}(Q, H^0(Y)) \end{aligned}$$

This implies that for any additive subcategory $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, we have that

$$\begin{aligned}
\mathcal{H}^{\perp_1} &= \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid \mathbb{E}(\mathcal{H}, X) = 0\} \\
&= \operatorname{add}\left(\left\{X_N \mid N \in {}^{\perp}\tau H^0(\mathcal{H}) \cap \big(\mathcal{H} \cap \operatorname{add}(\Lambda[1])\big)[-1]^{\perp}\right\} \cup \operatorname{add}(\Lambda[1])\right) \\
{}^{\perp_1}\mathcal{H} &= \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid \mathbb{E}(X, \mathcal{H}) = 0\} \\
&= \operatorname{add}\left(\left\{X_M \mid \tau M \in H^0(\mathcal{H})^{\perp}\right\} \cup \left(\operatorname{add}(\Lambda) \cap {}^{\perp}H^0(\mathcal{H})\right)[1]\right).
\end{aligned}$$

Now let $\mathcal{T} \subset \operatorname{mod}\Lambda$ be a torsion class and let $\mathcal{Y}_{\mathcal{T}} = \{X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda) \mid H^0(X) \in \mathcal{T}\}$. Since $\mathcal{T}$ is closed under direct summands and for every $X, Y \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, $H^0(X \oplus Y) \simeq H^0(X) \oplus H^0(Y)$, we readily see that $\mathcal{Y}_{\mathcal{T}}$ is additive. This implies that

$$ {}^{\perp_1}\mathcal{Y}_{\mathcal{T}} = \operatorname{add}\left(\left\{X_M \mid \tau M \in \mathcal{T}^{\perp}\right\} \cup \left(\operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}\right)[1]\right). $$

Hence

$$ H^0({}^{\perp_1}\mathcal{Y}_{\mathcal{T}}) = \{M \in \operatorname{mod}\Lambda \mid \tau M \in \mathcal{T}^{\perp}\} = \operatorname{add}\left(\tau^{-1}(\mathcal{T}^{\perp}) \cup \operatorname{add}(\Lambda)\right) $$

and

$$ {}^{\perp_1}\mathcal{Y}_{\mathcal{T}} \cap \operatorname{add}(\Lambda[1]) = \left(\operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}\right)[1]. $$

Finally, we get that

$$\begin{aligned}
&({}^{\perp_1}\mathcal{Y}_{\mathcal{T}})^{\perp_1} = \\
&\quad = \operatorname{add}\left(\left\{X_N \mid N \in {}^{\perp}\tau H^0({}^{\perp_1}\mathcal{Y}_{\mathcal{T}}) \cap \left({}^{\perp_1}\mathcal{Y}_{\mathcal{T}} \cap \operatorname{add}(\Lambda[1])\right)[-1]^{\perp}\right\} \cup \operatorname{add}(\Lambda[1])\right) \\
&\quad = \operatorname{add}\left(\left\{X_N \mid N \in {}^{\perp}\tau\left(\tau^{-1}(\mathcal{T}^{\perp}) \cup \operatorname{add}(\Lambda)\right) \cap \left(\operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}\right)^{\perp}\right\} \cup \operatorname{add}(\Lambda[1])\right)
\end{aligned}$$

The only thing left to prove is that

$$ {}^{\perp}\tau\left(\tau^{-1}(\mathcal{T}^{\perp}) \cup \operatorname{add}(\Lambda)\right) \cap \left(\operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}\right)^{\perp} = \mathcal{T}. $$

Note that $\tau\left(\tau^{-1}(\mathcal{T}^{\perp}) \cup \operatorname{add}(\Lambda)\right) = \tau\left(\tau^{-1}(\mathcal{T}^{\perp})\right)$. Since $\tau\left(\tau^{-1}(\mathcal{T}^{\perp})\right)$ and $\mathcal{T}^{\perp} \setminus (\mathcal{T}^{\perp} \cap \operatorname{inj}\Lambda)$ have the same indecomposables, we get that

$$ {}^{\perp}\tau\left(\tau^{-1}(\mathcal{T}^{\perp}) \cup \operatorname{add}(\Lambda)\right) = {}^{\perp}(\mathcal{T}^{\perp} \setminus (\mathcal{T}^{\perp} \cap \operatorname{inj}\Lambda)). $$

Given that $\mathcal{T}$ is a torsion class and hence $\mathcal{T} = {}^{\perp}(\mathcal{T}^{\perp})$, we have that

$$ \mathcal{T} \subset {}^{\perp}\left(\mathcal{T}^{\perp} \setminus (\mathcal{T}^{\perp} \cap \operatorname{inj}\Lambda)\right) \cap \left(\operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}\right)^{\perp}. $$

For the other inclusion, let M in ${}^{\perp}\left(\mathcal{T}^{\perp} \setminus (\mathcal{T}^{\perp} \cap \operatorname{inj}\Lambda)\right) \cap \left(\operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}\right)^{\perp}$. We want to prove that $M \in {}^{\perp}(\mathcal{T}^{\perp})$. Let L be an indecomposable in $\mathcal{T}^{\perp}$. If L is not injective, then $\operatorname{Hom}_{\Lambda}(M, L) = 0$ by definition of M. Assume thus that $L = I_i$, where $I_i \in \mathcal{T}^{\perp} \cap \operatorname{inj}\Lambda$ is the injective envelope of the simple S_i. Recall that for any $N \in \operatorname{mod}\Lambda$, $\operatorname{Hom}_{\Lambda}(N, I_i) = 0$ if and only if $\operatorname{Hom}_{\Lambda}(P_i, N) = 0$, where P_i is the projective cover of S_i. Thus $I_i \in \mathcal{T}^{\perp} \cap \operatorname{inj}\Lambda$ if and only if $\operatorname{Hom}_{\Lambda}(\mathcal{T}, I_i) = 0 = \operatorname{Hom}_{\Lambda}(P_i, \mathcal{T})$, which is equivalent to $P_i \in \operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}$. But $M \in \big(\operatorname{add}(\Lambda) \cap {}^{\perp}\mathcal{T}\big)^{\perp}$ as well, so we have that

$$\operatorname{Hom}_\Lambda(P_i, M) = 0 = \operatorname{Hom}_\Lambda(M, I_i).$$

We conclude that $M \in {}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$. In particular we get that

$$({}^{\perp_1}\mathcal{Y}_\mathcal{T})^{\perp_1} = \operatorname{add}\left(\{X_N \mid N \in \mathcal{T}\ \} \cup \operatorname{add}(\Lambda[1])\right) = \mathcal{Y}_\mathcal{T}$$

which implies that $({}^{\perp_1}\mathcal{Y}_\mathcal{T}, \mathcal{Y}_\mathcal{T})$ is a cotorsion pair. □

Theorem 4.44 induces a "mirror" of Theorem 4.2 in the category of projective presentations:

Corollary 4.45. *Let Λ be a finite-dimensional $\mathbb{k}$-algebra. The following are equivalent:*

1. *Λ is g-finite.*
2. *There exist finitely many complete cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.*
3. *All cotorsion pairs in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ are complete.*

Proof. The implications follow from Theorem 4.44, [PZ23, Theorem 3.7] and Theorem 4.2. □

4.3.2 All thick subcategories have enough injectives

Recall that functorially finite torsion classes are in bijection with left finite wide subcategories. Given that left wide subcategories are defined as those such that the smallest torsion class containing them is functorially finite, we have the following result.

Corollary 4.46. *If Λ is a g-finite finite-dimensional $\mathbb{k}$-algebra, then all wide subcategories are left finite.*

In Section 3.1.1, we introduced new maps between cotorsion pairs, thick subcategories and silting complexes in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. We showed that they mirror the maps between torsion classes, wide subcategories and support τ-tilting pairs, and that they restrict to bijections for certain subsets of these objects. In particular, we showed that thick subcategories with enough injectives of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ are the analogue of left finite wide subcategories. Unlike left finite wide subcategories, the defining characteristic of such thick subcategories is inherent to them. In this section, we establish an equivalence between being g-finite and having finitely many thick subcategories. Furthermore, we will show that if Λ is g-finite, then every thick subcategory in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is generated by a 2-term presilting complex, and that we can choose it to be injective in the thick subcategory it generates. First, we will need to show that if Λ is a g-finite algebra, then all non-trivial thick subcategories in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ contain a non-zero presilting complex. For any $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, we denote by $[X]$ the class of X in the Grothendieck group $K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$.

Proposition 4.47. *Let Λ be a finite-dimensional $\mathbb{k}$-algebra. Suppose that Λ is g-finite and let $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be a full additive subcategory that is closed under extensions. If $X \in \mathcal{H}$, then $\mathcal{H}$ contains a presilting object $0 \neq U$ such that $[X] = [U]$ or $\mathcal{H}$ contains a non-zero $P \in \operatorname{proj}\Lambda$ and its shift $P[1]$.*

To establish 4.47, we will employ an algebraic-geometric result concerning the varieties of bounded complexes of projective modules over a finite-dimensional algebra. Let $p \leq q \in \mathbb{Z}$ and consider $\mathcal{C}^{[p,q]}(\operatorname{proj}\Lambda) \subset \mathcal{C}^{b}(\operatorname{proj}\Lambda)$ the category of complexes of projective Λ-modules concentrated in degrees $[p,q]$. Fix a set of representatives $\{P_i\}_{1\leq 1 \leq n}$ of the isoclasses of indecomposable projective Λ-modules, then for any choice of $\bar{l} = (l_p, \cdots, l_{p+j}, \cdots, l_q) \in (\mathbb{Z}^n)^{q-p+1}$ where $l_j = (l_{j,1}, l_{j,2}, \cdots, l_{j,n})$, we define $R_{\bar{l}}$ to be the closed subvariety

$$R_{\bar{l}} \subset \prod_{j=0}^{q-p-1} \operatorname{Hom}_{\Lambda}\left(\bigoplus_{i=1}^{n} P_i^{\oplus l_{p+j,i}}, \bigoplus_{i=1}^{n} P_i^{\oplus l_{p+j+1,i}}\right)$$

defined by the relation $f_{p+i+1} \circ f_{p+i} = 0$ for all $0 \leq i \leq q-p-2$. In other words, $R_{\bar{l}}$ parametrizes all complexes in $\mathcal{C}^{[p,q]}(\operatorname{proj}\Lambda)$ with $\bigoplus_{i=1}^{n} P_i^{\oplus l_{p+j,i}}$ in position $p+j$. The variety $R_{\bar{l}}$ is equipped with a group action of

$$G_{\bar{l}} = \prod_{j=0}^{q-p} \operatorname{Aut}_{\Lambda}\left(\bigoplus_{i=1}^{n} P_i^{\oplus l_{p+j,i}}\right)$$

given by

$$(g_{p+j})_{0\leq j\leq q-p} \cdot (f_{p+i})_{0\leq i \leq q-p-1} = \left(g_{p+i+1} f_{p+i} g_{p+i}^{-1}\right)_{0\leq i\leq q-p-1}.$$

Theorem 4.48. *[JSZ05, Theorem 2] Let Λ be a finite-dimensional $\mathbb{k}$-algebra and suppose that that $\mathbb{k} = \bar{\mathbb{k}}$. Let $\bar{l} = (l_p, \cdots, l_{p+j}, \cdots, l_q) \in (\mathbb{Z}^n_{\geq 0})^{q-p+1}$ for some $p < q \in \mathbb{Z}$. Let $N, M \in R_{\bar{l}}$ such that $N \in \overline{G_{\bar{l}} \cdot M}$. Then there exists $m \in \mathbb{N}$ and exact sequences in $\mathcal{C}^b(\operatorname{proj}\Lambda)$*

$$0 \to N_0 \to N_{i+1} \to N_i \to 0$$

for any $0 \leq i \leq m-1$ such that $N_0 = N$ and $N_m \simeq M \oplus N'$ for some $N' \in \mathcal{C}^{[p,q]}(\operatorname{proj}\Lambda)$.

We will also make use of the following known result, which follows from [DIJ19].

Proposition 4.49. *[DIJ19] Suppose that Λ is g-finite. Then for any $\theta \in K_0(\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda))$ there exists a presilting object $X \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $[X] = \theta$.*

Theorem 4.50. *[DIJ19, Theorem 6.5] Let Λ be a finite-dimensional algebra and let U and V be 2-term presilting complexes in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then $[U] = [V]$ if and only if $U \simeq V$.*

Proof of Proposition 4.47. Note that if $\mathcal{H} = \{0\}$ then the propositions follows immediately. Suppose then that there is $0 \neq X \in \mathcal{H}$.

$\boxed{\textit{Case } [X] = 0\textit{:}}$ If $[X] = 0$, given that the only 2-term presilting complex with zero g-vector is 0, we have to show that there exists a non-zero projective module P such that $\begin{array}{c} P \\ \downarrow \\ 0 \end{array} \in \mathcal{H}$. Since $[X] = 0$, there exists $P \in \operatorname{proj}\Lambda$ such that $X \simeq \begin{array}{c} P \\ \downarrow f \\ P \end{array}$.

If f is not radical, then X is isomorphic to some $\begin{array}{c} P' \oplus P'' \\ \downarrow \left(\begin{smallmatrix} f' & 0 \\ 0 & a \end{smallmatrix}\right) \\ P' \oplus P \end{array}$ with a invertible.

Since $\mathcal{H}$ is additive we can assume that $\begin{array}{c} P' \\ \downarrow f' \\ P' \end{array} \in \mathcal{H}$. Thus we can suppose that f is radical and hence there exists $m \geq 1$ such that $f^m = 0$. Consider the morphism $\delta \in \operatorname{Hom}_b(X, X[1])$ given by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P \arrow[d, "f"] \\
P \arrow[r, "1_P"] \arrow[d, "f"] & P \arrow[d] \\
P \arrow[r] & 0
\end{tikzcd}$$

Since f is radical, then $\delta \neq 0$ and its mapping cone $\begin{array}{c} P \oplus P \\ \downarrow{\scriptstyle \left(\begin{smallmatrix} -f & 0 \\ 1_P & f \end{smallmatrix}\right)} \\ P \oplus P \end{array}$ is isomorphic to the complex in the rightmost column of the following diagram

$$\begin{tikzcd}
P \oplus P \arrow[r, "\left(\begin{smallmatrix} 1_P & f \\ 0 & 1_P \end{smallmatrix}\right)", "\simeq"'] \arrow[d, "\left(\begin{smallmatrix} -f & 0 \\ 1_P & f \end{smallmatrix}\right)"'] & P \oplus P \arrow[d, "\left(\begin{smallmatrix} 0 & f^2 \\ 1_P & 0 \end{smallmatrix}\right)"'] \\
P \oplus P \arrow[r, "\simeq", "\left(\begin{smallmatrix} 1_P & f \\ 0 & 1_P \end{smallmatrix}\right)"'] & P \oplus P.
\end{tikzcd}$$

This implies that $X^{(1)} = \mathrm{Cone}(\delta)[-1] \simeq \begin{array}{c} P \\ \downarrow_{f^2} \\ P \end{array}$ belongs to the thick subcategory $\mathcal{H}$ since it is an self-extension of X. By repeating this argument we can construct objects $X^{(i)} \simeq \begin{array}{c} P \\ \downarrow_{f^{2i}} \\ P \end{array}$ for every $i \geq 1$. By choosing i such that $2i \geq m$ we conclude that $P \oplus P[1] \simeq \begin{array}{c} P \\ \downarrow_{0=f^{2i}} \\ P \end{array}$ is in $\mathcal{H}$, and since $\mathcal{H}$ is closed under direct summands, then both P and $P[1]$ are in $\mathcal{H}$.

$\boxed{\textit{Case } [X] \neq 0\textit{:}}$ We first suppose that $\Bbbk = \bar{\Bbbk}$. Let $X \in \mathcal{H}$ such that $[X] \neq 0$, then $X = \begin{array}{c} X^{-1} \\ \downarrow_{f} \\ X^0 \end{array}$, where $[X^{-1}] \neq [X^0]$. Recall that we can decompose the g-vector of X as $[X] = \theta_X^+ - \theta_X^-$, where $\theta_X^+ = (\max\{0, \theta_{X,i}\})_{1 \leq i \leq n}$ and $\theta_X^- = (\max\{0, -\theta_{X,i}\})_{1 \leq i \leq n}$. By Proposition 4.49 there exist a 2-term presilting complex U whose g-vector is $[X]$. Take u to be the point in $\mathrm{Hom}_\Lambda(P^{\theta_X^-}, P^{\theta_X^+})$ corresponding to U, which has an open dense orbit (see for instance [Pla13, Lemma 2.16]). Choose $Q \in \operatorname{proj}\Lambda$ such that $P^{\theta_X^-} \oplus Q = X^{-1}$ and $P^{\theta_X^+} \oplus Q = X^0$, then $u \oplus 1_Q$ still has an open dense orbit in $\mathrm{Hom}_\Lambda(X^{-1}, X^0)$. Hence $\overline{G \cdot u \oplus 1_Q} = \mathrm{Hom}_\Lambda(X^{-1}, X^0)$, and $X \in \overline{G \cdot U \oplus \begin{array}{c} Q \\ \| \\ Q \end{array}}$. Since $\Bbbk$ is algebraically closed, Theorem 4.48 implies that $U \oplus \begin{array}{c} Q \\ \| \\ Q \end{array}$ can be constructed as a direct summand of a sequence of self-extensions of X in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$. In particular, U is contained in $\mathcal{H}$ and $[U] = [X]$. This proves the result over an algebraically closed field.

Now let $\Bbbk$ be any field and let $\mathbb{K} = \bar{\Bbbk}$. We have a fully faithful functor $\mathcal{C}^b(\operatorname{proj}\Lambda) \otimes_\Bbbk \mathbb{K} \hookrightarrow \mathcal{C}^b(\operatorname{proj}\Lambda \otimes_\Bbbk \mathbb{K})$ induced by $- \otimes_\Bbbk \mathbb{K}$. Let $X \in \mathcal{H}$ be as before and consider $\bar{X} = X \otimes_\Bbbk \mathbb{K} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda \otimes_\Bbbk \mathbb{K})$. By the previous argument we can find a 2-term presilting complex $\bar{U} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda \otimes_\Bbbk \mathbb{K})$ that is a direct summand of an object obtained by a sequence of self-extensions of $\bar{X}$. Explicitly, there are exact sequences in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda\otimes_{\Bbbk}\mathbb{K})$

$$\begin{tikzcd}
0 \arrow[r] & \bar{X}_0^{-1} \arrow[r] \arrow[d, "\bar{f}_0"] & \bar{X}_{i+1}^{-1} \arrow[r] \arrow[d, "\bar{f}_{i+1}"] & \bar{X}_i^{-1} \arrow[r] \arrow[d, "\bar{f}_i"] & 0 \\
0 \arrow[r] & \bar{X}_0^{0} \arrow[r] & \bar{X}_{i+1}^{0} \arrow[r] & \bar{X}_i^{0} \arrow[r] & 0
\end{tikzcd} \tag{4.3.1}$$

for $0\leq i\leq m-1$ where $\bar{X}=\begin{array}{c}X^{-1}\otimes_{\Bbbk}\mathbb{K}\\ \downarrow f\otimes_{\Bbbk}\mathbb{K}\\ X^0\otimes_{\Bbbk}\mathbb{K}\end{array}=\begin{array}{c}\bar{X}_0^{-1}\\ \downarrow\bar{f}_0\\ \bar{X}_0^{0}\end{array}$ and such that $\bar{U}$ is a direct summand of $\bar{X}_m=\begin{array}{c}\bar{X}_m^{-1}\\ \downarrow\bar{f}_m\\ \bar{X}_m^{0}\end{array}$. In particular, for $i=0$ we get

$$\begin{tikzcd}
0 \arrow[r] & \bar{X}_0^{-1} \arrow[r] \arrow[d, "\bar{f}_0"] & \bar{X}_{1}^{-1} \arrow[r] \arrow[d, "\bar{f}_{1}"] & \bar{X}_0^{-1} \arrow[r] \arrow[d, "\bar{f}_0"] & 0 \\
0 \arrow[r] & \bar{X}_0^{0} \arrow[r] & \bar{X}_{1}^{0} \arrow[r] & \bar{X}_0^{0} \arrow[r] & 0\ .
\end{tikzcd} \tag{4.3.2}$$

By definition, both rows in the commutative diagram 4.3.2 are short exact sequences. Since $\bar{X}_0^{-1}=X_0^{-1}\otimes_{\Bbbk}\mathbb{K}$ and $\bar{X}_0^{0}=X_0^{0}\otimes_{\Bbbk}\mathbb{K}$ are projective modules in $\operatorname{proj}\Lambda\otimes_{\Bbbk}\mathbb{K}$, both exact sequences split. In particular

$$\begin{aligned}\bar{X}_1^{-1}&\simeq\left(X_0^{-1}\otimes_{\Bbbk}\mathbb{K}\right)\oplus\left(X_0^{-1}\otimes_{\Bbbk}\mathbb{K}\right)\simeq\left(X_0^{-1}\oplus X_0^{-1}\right)\otimes_{\Bbbk}\mathbb{K}\\ \bar{X}_1^{0}&\simeq\left(X_0^{0}\otimes_{\Bbbk}\mathbb{K}\right)\oplus\left(X_0^{0}\otimes_{\Bbbk}\mathbb{K}\right)\simeq\left(X_0^{0}\oplus X_0^{0}\right)\otimes_{\Bbbk}\mathbb{K}\end{aligned}$$

and thus

$$\begin{aligned}\bar{f}_1&\in\operatorname{Hom}_{\Lambda\otimes_{\Bbbk}\mathbb{K}}\left(\left(X_0^{-1}\oplus X_0^{-1}\right)\otimes_{\Bbbk}\mathbb{K},\left(X_0^{0}\oplus X_0^{0}\right)\otimes_{\Bbbk}\mathbb{K}\right)\\ &\simeq\operatorname{Hom}_{\Lambda}\left(X_0^{-1}\oplus X_0^{-1},X_0^{0}\oplus X_0^{0}\right)\otimes_{\Bbbk}\mathbb{K}.\end{aligned}$$

We deduce that the exact sequence of complexes 4.3.2 is induced by an exact sequence of complexes in $\mathcal{C}^{[-1,0]}(\operatorname{proj}\Lambda)$ under the functor $-\otimes_{\Bbbk}\mathbb{K}$. By applying the same argument for every $0\leq i\leq m-1$ and its respective exact sequence 4.3.1, we deduce that all self-extensions of $\bar{X}=X\otimes_{\Bbbk}\mathbb{K}$ lie in $\mathcal{C}^{b}(\operatorname{proj}\Lambda)\otimes_{\Bbbk}\mathbb{K}$. In particular, there exists $U\in\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ which is a direct summand of a sequence of self-extensions of X such that $\bar{U}\simeq U\otimes_{\Bbbk}\mathbb{K}$. Moreover, U is presilting if and only if $\bar{U}$ is (see for instance [DIJ19, Propositions 6.6 b)]), and $[X]=[\bar{X}]=[\bar{U}]=[U]$, which finishes the proof. □

Theorem 4.51. *Let Λ be a g-finite, finite-dimensional $\Bbbk$-algebra. Let $\mathcal{H}$ be any thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, then there exists a presilting complex $U\in\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $\mathcal{H}=thick_{[-1,0]}(U)$.*

Proof. Let Λ be any g-finite finite-dimensional $\Bbbk$-algebra and let $\mathcal{H}$ be a thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. If $\mathcal{H}=\{0\}$ then we are done. If not, take $0\neq X\in\mathcal{H}$. By Proposition 4.47 there exists a presilting $0\neq U$ in $\mathcal{H}$ and we let $\operatorname{thick}_b(U)$ be the thick subcategory of $\mathcal{K}^{b}(\operatorname{proj}\Lambda)$ generated by U. By Lemma 4.34 we know that $\operatorname{thick}_{[-1,0]}(U)=\operatorname{thick}_b(U)\cap\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Consider $\mathcal{J}_U=\mathcal{K}^{b}(\operatorname{proj}\Lambda)/\operatorname{thick}_b(U)$ and $\rho:\mathcal{K}^{b}(\operatorname{proj}\Lambda)\to\mathcal{J}_U$ as in Theorem 4.36. By Theorem 4.43, we know that $\mathcal{H}'=\rho(\mathcal{H})$ is a thick subcategory of $\mathcal{J}_U\simeq\operatorname{per}^{[-1,0]}(C_U)$.

Now consider the functor

$$p_* : \operatorname{per}^{[-1,0]}(C_U) \to \mathcal{K}^{[-1,0]}(\operatorname{proj} H^0(C_U))$$

induced by the canonical projection $p : C_U \to H^0(C_U)$ as in Proposition 4.18. This functor induces an equivalence $p_* : \operatorname{per}^{[-1,0]}(C_U)/\mathcal{I} \to \mathcal{K}^{[-1,0]}(\operatorname{proj} H^0(C_U))$ where $\mathcal{I}$ is the ideal of morphisms that factor trough a morphism $X[1] \to Y$ where $X, Y \in \operatorname{add}(C_U)$. By Corollary 4.21, $p_*(\mathcal{H}')$ is closed under extensions and direct summands. Moreover, if $\mathcal{H}' \neq \{0\}$ then $p_*(\mathcal{H}') \neq \{0\}$ since p_* preserves isomorphism classes. As recalled in Remark 4.39, $\operatorname{mod}(H^0(C_U))$ is equivalent to the τ-tilting reduction of $\operatorname{mod}\Lambda$ associated to U introduced in Section 4.2.1, in particular $H^0(C_U)$ is g-finite by Theorem 4.31. By applying Proposition 4.47 to the category $p_*(\mathcal{H}') \subset \mathcal{K}^{[-1,0]}(\operatorname{proj} H^0(C_U))$, there exists a 2-term presilting object $0 \neq V' \in p_*(\mathcal{H}')$, and thus there exists $V \in \mathcal{H}'$ such that $V' = p_*(V)$ which is itself presilting by Proposition 4.22. Moreover, $V \in \rho(\mathcal{H}) \simeq \bar{\rho}(\mathcal{H} \cap \mathcal{Z}_{\mathcal{U}})$, thus by Theorem 4.37 we know that $V = W \oplus U \in \mathcal{H} \cap \mathcal{Z}_{\mathcal{U}} \subset \mathcal{H}$, where $W \neq 0$ since we supposed V' and thus V to be non-zero in $\mathcal{J}_U$. By substituting U by V in the previous argument, we can find a sequence of 2-term presilting complexes $(V_i)_{i \in \mathbb{N}}$ such that $\operatorname{add}(V_i) \subsetneq \operatorname{add}(V_{i+1})$. Since the number of indecomposables of a presilting 2-term complex is bounded by $|\Lambda|$, the sequence must stabilize, that is for $i \gg 0$

$$\operatorname{add}(V_i) = \operatorname{add}(V_{i+1}) = \operatorname{add}(\bar{U}),$$

where $\bar{U} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ is a basic presilting complex. Then $\mathcal{H} = \operatorname{thick}_{[-1,0]}(\bar{U})$, which gives the result. □

Corollary 4.52. *Let Λ be a finite-dimensional algebra, then Λ is g-finite if and only if there exist finitely many thick subcategories in $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$.*

Proof. Suppose Λ is g-finite. By Theorem 4.51, any thick subcategory is generated by a presilting complex $U \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, and since there are finitely many isomorphism classes of such U, then there exist finitely many thick subcategories. Conversely, if there are finitely many thick subcategories, then there are also finitely many of them with enough injectives. Given that Corollary 3.33 establishes a bijection between isomorphism classes 2-term silting complexes and thick subcategories with enough injectives, we conclude that Λ is g-finite.

□

Proposition 4.53. *Let $\mathcal{C} \subset \operatorname{mod}\Lambda$ and $\mathcal{H} \subset \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ be subcategories, then*

$$\mathscr{W}(\mathcal{H}) = \mathcal{H}^{\perp_{\mathbb{Z}}} \cap \operatorname{mod}\Lambda$$
$$\mathscr{T}(\mathcal{C}) = {}^{\perp_{\mathbb{Z}}}\mathcal{C} \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$$

where

$$\mathcal{H}^{\perp_{\mathbb{Z}}} = \{Y \in \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(\mathcal{H}, Y[i]) = 0 \ \forall\ i \in \mathbb{Z}\}$$
$${}^{\perp_{\mathbb{Z}}}\mathcal{C} = \{Y \in \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(Y[i], \mathcal{C}) = 0 \ \forall\ i \in \mathbb{Z}\}$$

and $\mathcal{D} = \mathcal{D}^b(\operatorname{mod}\Lambda)$.

Proof. Let $M \in \operatorname{mod}\Lambda$ and $X = \begin{smallmatrix} X^{-1} \\ \downarrow f \\ X^0 \end{smallmatrix} \in \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Then $M \in \mathscr{W}(X)$ (or equivalently $X \in \mathscr{T}(M)$) if and only if the $\Bbbk$-linear map

$$\operatorname{Hom}_\Lambda(X^0, M) \xrightarrow{\operatorname{Hom}_\Lambda(f,M)} \operatorname{Hom}_\Lambda(X^{-1}, M)$$

is an isomorphism. By definition, $X \in {}^\perp M[i]$ (or equivalently $M \in X[-i]^\perp$) for any $i \neq 0, 1$, so the only thing left to prove is the case when $i = 0$ or 1. Consider the triangle

$$X[-1] \dashrightarrow X^{-1} \xrightarrow{f} X^0 \to X.$$

Then by applying $\operatorname{Hom}_\mathcal{D}(-, M)$ we get an exact sequence

$$\begin{tikzcd}
\operatorname{Hom}_\mathcal{D}(X,M) \arrow[r] & \operatorname{Hom}_\mathcal{D}(X^0,M) \arrow[r, "\operatorname{Hom}_\mathcal{D}(f,M)"] \arrow[d, equal] & \operatorname{Hom}_\mathcal{D}(X^{-1},M) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_\mathcal{D}(X[-1],M) \\
 & \operatorname{Hom}_\Lambda(X^0,M) \arrow[r, "\operatorname{Hom}_\Lambda(f,M)"] & \operatorname{Hom}_\Lambda(X^{-1},M) &
\end{tikzcd}$$

This implies that $\operatorname{Hom}_\Lambda(f, M)$ is an isomorphism if and only if $\operatorname{Hom}_\mathcal{D}(X[-1], M) \simeq 0 \simeq \operatorname{Hom}_\mathcal{D}(X, M)$, which is equivalent to $M \in X^{\perp_\mathbb{Z}}$ (and $X \in {}^{\perp_\mathbb{Z}}M$).

□

Lemma 4.54. *Let Γ be a non-positive dg algebra over $\Bbbk$ such that $H^0(\Gamma)$ is finite-dimensional. Let S be a simple module in the heart of the standard t-structure in $\mathcal{D}(\Gamma)$. Then $X \in \operatorname{per}(\Gamma)$ belongs to ${}^{\perp_\mathbb{Z}}S$ if and only if $X \in \operatorname{thick}_{\operatorname{per}\Gamma}(\operatorname{add}(\Gamma)\backslash\operatorname{add}(P))$ where P is the direct summand of Γ satisfying that $\operatorname{Hom}_{\mathcal{D}(\Gamma)}(P, S[i])$ is a division algebra if $i = 0$ and 0 otherwise.*

Proof. We are going to prove that if $X \in {}^{\perp_\mathbb{Z}}S \cap \operatorname{per}\Gamma$ then $X \in \operatorname{thick}_{\operatorname{per}\Gamma}(\operatorname{add}(\Gamma) \setminus \operatorname{add}(P))$. The other direction is straightforward. Let $X \in \operatorname{per}\Gamma$, then by [Pla11, Lemma 2.14] X is quasi-isomorphic to a *twisted complex* in $\mathcal{D}(\Gamma)$, that is, its underlying graded module is of the form

$$\bigoplus_{i=1}^{l} Q_i$$

where $Q_i \in \operatorname{add}(\Gamma)[n_i]$ for some $n_i \in \mathbb{Z}$ and its differential is given by

$$\begin{pmatrix} d_1 & f_{12} & \cdots & f_{1l} \\ 0 & d_2 & \cdots & f_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_l \end{pmatrix}$$

where d_i is the differential corresponding to Q_i. Recall that for any $Q \in \operatorname{add}(\Gamma)[n]$, then $\operatorname{Hom}_{\mathcal{D}(\Gamma)}(Q, S[i]) = 0$ if and only if $Q \notin \operatorname{add}(P)$ or $i \neq n$. Since $\operatorname{Hom}_{\mathcal{D}(\Gamma)}(X, S[i]) = 0$ for all $i \in \mathbb{Z}$, we conclude that none of the Q_l are shifts of P. In particular, $X \in \operatorname{thick}_{\operatorname{per}\Gamma}(\operatorname{add}(\Gamma) \setminus \operatorname{add}(P))$.

□

Proposition 4.55. *Let*

$$\begin{aligned}\mathscr{W} &: \operatorname{thick}\Lambda \to \operatorname{wide}\Lambda\\ \mathscr{T} &: \operatorname{wide}\Lambda \to \operatorname{thick}\Lambda\end{aligned}$$

be the maps defined in Proposition 4.53. Then $\mathscr{W}$ and $\mathscr{T}$ induce mutually inverse bijections between the set of left finite wide subcategories in $\operatorname{mod}\Lambda$ *and thick subcategories with enough injectives in* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$*.*

Proof. Let $\mathcal{H} = \operatorname{thick}_{[-1,0]}(U)$ where U a basic injective generator of $\mathcal{H}$. Then by Proposition 4.53 we know that $\mathscr{W}(\mathcal{H}) = \mathscr{W}(U) = U^{\perp_{\mathbb{Z}}} \cap \operatorname{mod}\Lambda$. By Theorem 3.72, we know that $\mathscr{W}(\mathcal{H}) = \mathcal{W}_{(M,P)}$ where (M,P) is the support τ-tilting pair associated to the presilting complex U. The only thing we need to prove is that $\mathscr{T}(\mathcal{W}_{(M,Q)}) \subset \operatorname{thick}_{[-1,0]}(U)$, since the other inclusion is always satisfied.

Recall that since U is a 2-term presilting complex, it can be completed into a 2-term silting complex. Let T_U the Bongartz completion of U. Since U is injective in $\mathcal{H} = \operatorname{thick}_{[-1,0]}(U)$, then $T_U = U \oplus V$ where V is the Bongartz complement of U. Let $\mathcal{S} \subset \mathcal{D}^{[-1,0]}(\operatorname{mod}\Lambda)$ be the simple-minded collection associated to T_U, then $\mathcal{S} = \mathcal{S}_U \sqcup \mathcal{S}_V$ where $\mathcal{S}_U = U^{\perp_{\mathbb{Z}}} \cap \mathcal{S}$ and $\mathcal{S}_V = V^{\perp_{\mathbb{Z}}} \cap \mathcal{S}$. By Theorem 4.25, we know that $\mathcal{S}_U \subset \operatorname{mod}\Lambda$ and that $\mathscr{W}(U) = \mathcal{W}_{(M,Q)} = \operatorname{Filt}(\mathcal{S}_U)$. Thus $\mathscr{T}(\mathscr{W}(\mathcal{H})) = {}^{\perp_{\mathbb{Z}}}\mathcal{S}_U \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. Recall that there exists a non-positive dg algebra Γ_U and triangulated equivalences

$$\begin{tikzcd}
\mathcal{D}_{fd}(\Gamma_U) \arrow[r, "\simeq", "\phi"'] & \mathcal{D}^b(\operatorname{mod}\Lambda) \\
\operatorname{per}(\Gamma_U) \arrow[r, "\simeq", "\bar{\phi}"'] \arrow[u, hook] & \mathcal{K}^b(\operatorname{proj}\Lambda) \arrow[u, hook]
\end{tikzcd}$$

which take Γ_U to T_U and all simple $H^0(\Gamma_U)$-modules to the simple-minded collection $\mathcal{S}$ (Theorem 4.17). Let $\bar{U} \in \operatorname{add}(\Gamma_U)$ be the perfect complex sent to U and $\overline{\mathcal{S}_U}$ the set of simple $H^0(\Gamma_U)$-modules that are perpendicular to $\bar{U}$. Then $\phi(\overline{\mathcal{S}_U}) = \mathcal{S}_U$ and thus

$$\begin{aligned}\mathscr{T}(\mathscr{W}(\mathcal{H})) &= {}^{\perp_{\mathbb{Z}}}\mathcal{S}_U \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)\\ &= {}^{\perp_{\mathbb{Z}}}\phi\left(\overline{\mathcal{S}_U}\right) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)\\ &= \phi\left({}^{\perp_{\mathbb{Z}}}\overline{\mathcal{S}_U}\right) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)\end{aligned}$$

By Lemma 4.54, we know that ${}^{\perp_{\mathbb{Z}}}\overline{\mathcal{S}_U} = \operatorname{thick}_{\operatorname{per}(\Gamma_U)}(\bar{U})$. Since ϕ is a triangle equivalence $\phi(\operatorname{thick}_{\operatorname{per}(\Gamma_U)}(\bar{U})) = \operatorname{thick}_b(U)$. This implies that $\mathscr{T}(\mathscr{W}(\mathcal{H})) = \operatorname{thick}_b(U) \cap \mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$. By applying Lemma 4.34 we get that

$$\mathscr{T}(\mathscr{W}(\mathcal{H})) = \operatorname{thick}_{[-1,0]}(U) = \mathcal{H}.$$

□

Theorem 4.56. *Suppose Λ is g-finite. Then all thick subcategories of* $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ *have enough injectives.*

Proof. Let $\mathcal{H}$ be a thick subcategory of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$, by Theorem 4.51 we know that there exists a 2-term presilting complex U such that $\mathcal{H} = \operatorname{thick}_{[-1,0]}(U)$. Since Λ is g-finite, then $\mathscr{W}(\mathcal{H})$ is left finite and by Proposition 4.55 $\mathscr{T}(\mathscr{W}(\mathcal{H}))$ has enough injectives. Let V be an injective generator of $\mathscr{T}(\mathscr{W}(\mathcal{H}))$, then

$$\operatorname{thick}_{[-1,0]}(U) \subset \mathscr{T}(\mathscr{W}(\operatorname{thick}_{[-1,0]}(U))) = \operatorname{thick}_{[-1,0]}(V).$$

Since $U \in \operatorname{thick}_{[-1,0]}(V) = \operatorname{Cocone}(V, V)$ [Gar23, Lemma 3.9], there exists a conflation $U \rightarrowtail V' \twoheadrightarrow V''$ with $V', V'' \in \operatorname{add}(V)$ and thus U is 2-term presilting in $\operatorname{thick}_b(V)$ with respect to $V[-1]$. By Lemma 4.13, we know that U can be completed into a silting complex in $\operatorname{thick}_b(V)$. Since U and V give rise to the same τ-perpendicular category $\mathscr{W}(\mathcal{H})$, we deduce that $|U| = |V| = |V[-1]|$, which in turn implies by Proposition 4.14 that U is already silting in $\operatorname{thick}_b(V)$ and thus $\operatorname{thick}_b(V) = \operatorname{thick}_b(U)$. By Corollary 4.35 this implies that

$$\mathcal{H} = \operatorname{thick}_{[-1,0]}(U) = \operatorname{thick}_{[-1,0]}(V).$$

□

Remark 4.57. The arguments used both in Proposition 4.55 and Theorem 4.56 can be adapted to show that the results hold if we substitute left finite for right finite wide subcategories introduced in Section 4.1.5. That is, $\mathscr{W}$ and $\mathscr{T}$ are inverse of each other if restricted to the set of right finite wide subcategories and thick subcategories with enough projectives. Moreover, if Λ is g-finite, the dual statement of Theorem 4.56 also holds.

Theorem 4.58. *Suppose Λ is g-finite. Then all thick subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ have enough projectives.*

Corollary 4.59. *Suppose Λ is g-finite. Then all thick subcategories of $\mathcal{K}^{[-1,0]}(\operatorname{proj}\Lambda)$ have enough projectives and enough injectives.*

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